Math 30820 Honors Algebra 4
Homework 6

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Due Wednesday, 3/4/2020

Do 5.

1. Show that every extension K/F with $[K:F]=2$ is a normal extension. When is such an extension separable?

2. Let $P \in F[X]$, of degree n, and K be the splitting field of P over F. Show that $[K:F] \mid n!$.

3. Let F be a field, $P \in F[X]$ a *monic* polynomial and K a field that contains all the roots $\alpha_1, \ldots, \alpha_n$ of the polynomial $P(X)$, where n is the degree of $P(X)$. The **discriminant** of $P(X)$ is defined as

$$\Delta = \prod_{1 \le i < j \le n} (\alpha_i - \alpha_j)^2$$

   Show that P is separable if and only if $\Delta \neq 0$ and that

$$\Delta = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} P'(\alpha_i)$$

4. (Do one of the 2 parts)

   (a) Consider the polynomial $P(X) = X^5 + pX + q$. Show that it has discriminant

$$\Delta = 5^5 q^4 + 4^4 p^5$$

   (b) (This part is worth 2 extra points) Consider the polynomial $P(X) = X^n + pX + q$. Show that it has discriminant

$$\Delta = (-1)^{\binom{n}{2}} n^n q^{n-1} + (-1)^{\binom{n-1}{2}} (n-1)^{n-1} p^n$$

5. Let $\alpha \in \mathbb{R}$ such that $\alpha^4 = 5$.

   (a) Is $\mathbb{Q}(i\alpha^2)$ normal over $\mathbb{Q}$?

   (b) Is $\mathbb{Q}(\alpha + i\alpha)$ normal over $\mathbb{Q}(i\alpha^2)$?

   (c) Is $\mathbb{Q}(\alpha + i\alpha)$ normal over $\mathbb{Q}$?

6. Let F be a field of characteristic p that is not perfect, i.e., the Frobenius homomorphism $\phi : F \to F$ given by $\phi(x) = x^p$ is not surjective. Show that there exist inseparable irreducible polynomials in $F[X]$.

7. Let F be a field of characteristic p and let K/F be a finite extension with $p \nmid [K:F]$. Show that K/F is a separable extension, i.e., for every $\alpha \in K$ the minimal polynomial of α over F is a separable polynomial.