Math 30820 Honors Algebra 4
Homework 8

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Due Wednesday, 4/1/2020

Do 4.

Throughout this problem set $\Phi_n(X)$ is the n-th cyclotomic polynomial.

1. For a positive integer n we denote by $s(n)$ the largest square-free divisor of n. Show that

$$\Phi_n(X) = \Phi_{s(n)}(X^{n/s(n)})$$

[Hint: Use the Möbius inversion formula.]

2. Show that

$$\Phi_n(1) = \begin{cases} 0 & n = 1 \\ p & n = p^a \\ 1 & n = p_1^{a_1} \cdots p_k^{a_k}, k \geq 2 \end{cases}$$

[Hint: Use induction.]

3. Show that

$$\prod_{1 \leq k \leq n, (k,n)=1} \sin\left(\frac{k\pi}{n}\right) = \frac{\Phi_n(1)}{2^{\varphi(n)}}$$

where φ is Euler's function. Remark that $\Phi_n(1)$ is computed in the previous problem. [Hint: Write $\Phi_n(1)$ as a product over the primitive roots of 1 and use double angle formulas.]

4. Let p be a prime. Let F be the union of the fields of rational functions $\mathbb{F}_p(x) \subset \mathbb{F}_p(\sqrt[p]{x}) \subset \mathbb{F}_p(\sqrt[p^2]{x}) \subset \ldots \subset \mathbb{F}_p(\sqrt[p^n]{x}) \subset \ldots$. Show that F is the smallest perfect field containing $\mathbb{F}_p(x)$.

5. Suppose $\sigma : K \to L$ is a field isomorphism which sends the subfield $F \subset K$ to the subfield $\sigma(K) \subset L$. Show that $\mathrm{Aut}(L/\sigma(F)) = \sigma \,\mathrm{Aut}(K/F)\sigma^{-1}$.

6. Let K be the splitting field over $\mathbb{Q}$ of $X^8 - 2$. Show that $\mathrm{Gal}(K, \mathbb{Q}(i)) \cong \mathbb{Z}/8\mathbb{Z}$ and $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{2})) \cong D_4$, the dihedral group with 8 elements.

7. Let $\alpha_1 = \sqrt{1 + \sqrt{3}}$, $\alpha_2 = \sqrt{1 - \sqrt{3}}$, two roots of the irreducible polynomial $X^4 - 2X^2 - 2 \in \mathbb{Q}[X]$.
   (a) Show that $\mathbb{Q}(\alpha_1) \cap \mathbb{Q}(\alpha_2) = \mathbb{Q}(\sqrt{3})$.
   (b) Show that $\mathbb{Q}(\alpha_1)$, $\mathbb{Q}(\alpha_2)$ and $\mathbb{Q}(\alpha_1, \alpha_2)$ are Galois over $\mathbb{Q}(\sqrt{3})$ and that $\mathrm{Gal}(\mathbb{Q}(\alpha_1, \alpha_2)/\mathbb{Q}(\sqrt{3})) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

8. Show that $K = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$ is Galois over $\mathbb{Q}$ and that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$.