Math 30820 Honors Algebra 4
Homework 9

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Do 4.

1. Recall from class that if L is the splitting field of $X^8 - 2$ over $\mathbb{Q}$ then we embedded $\mathrm{Gal}(L/\mathbb{Q})$ into group $\begin{pmatrix} (\mathbb{Z}/8\mathbb{Z})^\times & \mathbb{Z}/8\mathbb{Z} \\ 0 & 1 \end{pmatrix}$, an automorphism $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ being associated with a matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ if $\sigma(\zeta) = \zeta^a$ and $\sigma(\sqrt[8]{2}) = \zeta^b \sqrt[8]{2}$ and that in this case the following compatibility must occur:
$$\sigma(\zeta + \zeta^{-1}) = \sigma(\sqrt{2}) = \sigma(\sqrt[8]{2})^4.$$
Show that $\mathrm{Gal}(L/\mathbb{Q})$ can be identified with the subgroup of matrices $\{\begin{pmatrix} \pm 3 & \text{odd} \\ 0 & 1 \end{pmatrix}\} \cup \{\begin{pmatrix} \pm 1 & \text{even} \\ 0 & 1 \end{pmatrix}\}$. (Remark: this Galois group is the semidirect product $\mathbb{Z}/8\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ given by $\mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{Z}/8)$ sending 1 to multiplication by 3.)

2. Let $p > 2$ be a prime number. Show that $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p}) \subset \mathbb{Q}(\zeta_p)$. [Hint: Compute the discriminant of $X^p - 1$.]

3. Write a computer program that computes the discriminant of the polynomial
$$P(X) = X^5 + aX^4 + bX^3 + cX^2 + dX + e$$
as a polynomial in a, b, c, d, e. Submit the expression and the code.

4. Artin 16.1.1 on page 505.

5. Artin 16.2.2 on page 506.

6. Artin 16.2.7 on page 506.

7. Artin 16.M.7 on page 512.

8. Let $K = \mathbb{Q}(\zeta_7, \sqrt[7]{2})$ with Galois group $\mathrm{Gal}(K/\mathbb{Q}) \cong \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_7)\}$. Let $H \subset \mathrm{Gal}(K/\mathbb{Q})$ be the subgroup generated by the matrix $A = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$. Show that $K^H = \mathbb{Q}(\zeta_7^6 \sqrt[7]{2})$.