Math 30820 Honors Algebra 4
Homework 11

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Due Wednesday, 4/29/2020

Do 4.

Let $C_5 = \mathbb{Z}/5\mathbb{Z}$, D_5 the dihedral group with 10 elements, $F_{20} = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{F}_5)\} \cong \mathbb{Z}/5\mathbb{Z} \rtimes (\mathbb{Z}/5\mathbb{Z})^\times$. You may take for granted that every transitive subgroup of S_5 is isomorphic to one of the groups C_5, D_5, F_{20}, A_5 and S_5.

1. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $P(X) = X^5 - 5X + 12 \in \mathbb{Q}[X]$.
   (a) Show that $P(X)$ is irreducible.
   (b) Show that $10 \mid [K : \mathbb{Q}]$. [Hint: $P(X)$ has two pairs of complex conjugate roots.]
   (c) Assume that $P(X)$ is solvable by radicals (it is), which is equivalent to $\mathrm{Gal}(K/\mathbb{Q})$ is a solvable group. Show that $\mathrm{Gal}(K/\mathbb{Q}) \cong D_5$, the dihedral group with 10 elements. [Hint: What is the discriminant of $P(X)$?]
2. Artin 16.9.12 on page 509.
3. Artin 16.9.13 on page 509.
4. Artin 16.9.18 on page 509.
5. Artin 16.M.10 on page 512.
6. Let t be an indeterminate. Show that
$$\mathbb{F}_2(t)^{\mathrm{Aut}(\mathbb{F}_2(t)/\mathbb{F}_2)} = \mathbb{F}_2\left(\frac{(t^2+t+1)^3}{t^2(t+1)^2}\right).$$
7. Show that the irreducible polynomial $x^4 + 1 \in \mathbb{Z}[X]$ is reducible mod every prime. [Hint: Use $\mathbb{F}_{p^2}$.]