Math 30810 Honors Algebra 3
Homework 2

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Due Wed, September 15

Do 5 problems.

1. Artin 2.2.2 on page 69.
2. Artin 2.2.4 on page 70.
3. Artin 2.2.6 on page 70.
4. Artin 2.3.1 on page 70.
5. Let B be the subset of $\mathrm{GL}_n(\mathbb{R})$ consisting of upper-triangular matrices. Show that B is a subgroup of $\mathrm{GL}_n(\mathbb{R})$.
6. Show that the set of matrices

$$H = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_{n-1} & x_n & z \\ 0 & 1 & 0 & 0 & \dots & 0 & y_1 \\ 0 & 0 & 1 & 0 & \dots & 0 & y_2 \\ & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & y_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \mid x_1, \dots, x_n, y_1, \dots, y_n, z \in \mathbb{R} \right\}$$

   forms a subgroup of $\mathrm{GL}_{n+2}(\mathbb{R})$. It is called the Heisenberg group. (Yes, that Heisenberg.)

7. Suppose $P(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$. If $q = \dfrac{m}{n}$ is a rational root of $P(X)$, written in lowest terms, show that $m \mid a_0$ and $n \mid a_d$. Factor $6X^3 + 7X^2 - 14X - 15$ over $\mathbb{Q}$.
8. Suppose G is a commutative group and $x, y \in G$. Show that $\mathrm{ord}(xy) \mid [\mathrm{ord}(x), \mathrm{ord}(y)]$ and that equality must occur when $\mathrm{ord}(x)$ and $\mathrm{ord}(y)$ are coprime.
9. Let $\varphi(n)$ be the number of integers between 1 and n which are coprime to n. E.g., $\varphi(6) = 2$ because 1 and 5 are the only integers between 1 and 6 which are coprime to 6. Show that in any cyclic group $\langle x \rangle$ of size n there are $\varphi(d)$ elements of order d for any $d \mid n$, and conclude that

$$\sum_{d \mid n} \varphi(d) = n.$$