Math 30810 Honors Algebra 3
Homework 4

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Due Wednesday, September 29

Do 5.

1. Artin 2.6.7
2. Artin 2.8.4.
3. Artin 2.8.6.
4. Artin 2.8.9.
5. (Very important) Suppose $H < G$ such that $[G : H] = 2$. Show that $H \lhd G$.
6. Suppose H is a subgroup of a group G. Show that if H is a subgroup such that $gHg^{-1} \subset H$ for all $g \in G$, then $H \lhd G$.
7. Show that if G is a group of order 4 then either it is cyclic or it is isomorphic to the Klein 4-group V.
8. (This is a useful problem, in the textbook it's 2.5.6) For $1 \leq i, j \leq n$ consider the matrix $E_{ij} \in M_{n\times n}(\mathbb{C})$ with 1 in position ij and 0s everywhere else.
   (a) For $i \neq j$ show that $I_n + E_{ij} \in \mathrm{GL}_n(\mathbb{C})$.
   (b) For a general matrix $X \in \mathrm{GL}_n(\mathbb{C})$ compute XE_{ij} and $E_{ij}X$ and show that the only invertible matrices that commute with every other invertible matrix are the scalar ones.
9. Let $G = \mathrm{GL}_2(\mathbb{F}_2)$. Show that any $g \in G$ permutes the nonzero vectors of $\mathbb{F}_2^2$. If you order these 3 non-zero vectors, this gives a map $f : G \to S_3$. Show that this map is an isomorphism.