Math 30810 Honors Algebra 3
Homework 5

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Due Wednesday, October 13

Do 5.

1. Artin 2.9.5.

2. Artin 2.9.7.

3. Let G be a group and $g \in G$. Suppose $g^m = e$ and $g^n = e$ where m and n are coprime integers. Show that $g = e$.

4. Compute $12^{34^{56^{78}}} \mod 90$.

5. Let G be a group.

   (a) Assume that H and K are subgroups and $|H| = |K| = p$ is a prime number. Show that either $H = K$ or $H \cap K = \{e\}$.

   (b) Let G be a group and $H_1, \ldots, H_k$ be distinct subgroups of G. Suppose that each group H_i has order p, a fixed prime number. Show that $H_1 \cup \ldots \cup H_k$ has exactly $(p-1)k + 1$ elements.

6. Suppose G is a finite group and p is a prime number such that every element $g \in G - \{e\}$ has order p. Show that $p - 1 \mid |G| - 1$. [Hint: use exercise 5.]

7. Let $G = \mathrm{GL}_2(\mathbb{R})$ and B the subgroup of upper triangular matrices. Show that

$$\mathrm{GL}_2(\mathbb{R}) = B \sqcup B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B.$$

   [Hint: Midterm exercise and example from class.]

   Theorem 1 (Lifting the Exponent or LTE)**.** *For a prime p and an integer n we denote $v_p(n)$ the power of p in the factorization of n. E.g., $v_3(12) = 1$, $v_2(5/4) = -2$, etc. Suppose $a \equiv b \pmod p$ are two integers. Then*

$$v_p(a^n - b^n) = v_p(a - b) + v_p(n).$$

8. Use LTE to show that $\mathrm{ord}(p + 1 \mod p^n) = p^{n-1}$ for every odd prime p but $\mathrm{ord}(3 \mod 2^n) = 2^{n-2}$.

9. Suppose $p > 2$ is a prime. Let $a \in \mathbb{Z}$ be a generator of the cyclic group $(\mathbb{Z}/p\mathbb{Z})^\times$. Show that $a^{p^{n-1}}(1+p)$ is a generator of the (necessarily) cyclic group $(\mathbb{Z}/p^n\mathbb{Z})^\times$.