Math 30810 Honors Algebra 3
Homework 6

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Due Wednesday, October 16

Do 5.

1. Suppose p is a prime number and a an integer such that $p \nmid a$. Show that $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$, and that you get $\equiv 1 \pmod{p}$ precisely when $a \equiv x^2 \pmod{p}$ for some integer p. This value is called the Legendre symbol, denoted $\left(\frac{a}{p}\right)$.

2. Let p be a prime number, and a, n integers such that $p \nmid a$. Show that $x^n \equiv a \pmod{p}$ has either no solutions, or exactly $(n, p-1)$ solutions. [Hint: Use the fact that $(\mathbb{Z}/p\mathbb{Z})^\times$ has a cyclic generator.]

3. Consider the complex number $\zeta = e^{2\pi i/10}$ which generates the cyclic group $G = \langle \zeta \rangle$ of order 10. Show that the only homomorphisms $f : S_3 \to G$ are the trivial homomorphism and the sign homomorphism $\varepsilon(\sigma) \in \{-1, 1\}$. (Note that $\zeta^5 = -1$ so $\{-1, 1\} \subset \langle \zeta \rangle$.) [Hint: what is $f(A_3)$?]

4. Let $p \equiv 3 \pmod{4}$ be a prime number. Suppose $x^2 \equiv y \pmod{p}$. Show that $x \equiv \pm y^{(p+1)/4} \pmod{p}$.

5. I tell you that $x^2 \equiv 1521 \pmod{2021}$. What is $x \mod 2021$? [Feel free to use wolfram alpha for computations of the type $a^b \mod c$, but you're not allowed to use an exhaustive search. It's easier to use the Chinese Remainder Theorem and the previous exercise.]

6. Write out, in detail, the proof I did in class that $\mathrm{Aut}(S_3) \cong S_3$.

7. Suppose $m, n \geq 1$ are integers. Show that $\mathrm{Aut}((\mathbb{Z}/m\mathbb{Z})^n) \cong \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$, the group of invertible $n \times n$ matrices with entries in $\mathbb{Z}/m\mathbb{Z}$.

8. (This is the 2nd isomorphism theorem, a straightforward application of the 1st isomorphism theorem) Let H, N be subgroups of a group G such that $N \lhd G$.
   (a) Show that $HN = \{hn \mid h \in H, n \in N\}$ is a subgroup of G and that $N \lhd HN$.
   (b) Let $\iota : H \to HN$ be the inclusion map and $\pi : HN \to HN/N$ the usual projection map. Show that $\pi \circ \iota$ is a surjective group homomorphism.
   (c) Show that the kernel of this composition is $H \cap N$ and conclude that
   $$H/H \cap N \cong HN/N.$$
   (This will be important next semester, in computing Galois groups of composite extensions.)

9. (This is the 3rd isomorphism theorem, a straightforward application of the 1st isomorphism theorem) Let $N \lhd H \lhd G$.
   (a) Show that $H/N \lhd G/N$.
   (b) Let $\pi : G \to G/N$ and $\pi' : (G/N)/(H/N)$ be the usual projection maps. Show that $\pi' \circ \pi$ is a surjective group homomorphism.

(c) Show that the kernel of this composition is H and conclude that

$$G/H \cong (G/N)/(H/N).$$

(This will be important next semester, in the main theorem of Galois theory.)