# Math 30810 Honors Algebra 3
# Homework 7

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Due in class on Wednesday, November 3

**Do 5.**

1. Let $n \geq 3$ be an integer. The dihedral group $D_n$ with $2n$ elements is
$$D_n = \{1, R, R^2, \ldots, R^{n-1}, F, FR, \ldots, FR^{n-1}\}$$
satisfying $\mathrm{ord}(R) = n$, $\mathrm{ord}(F) = 2$ and $FRF = R^{n-1}$. Show that $D_n$, as described above, is a group. Moreover, show that it isomorphic to the subgroup of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ consisting of matrices of the form $\begin{pmatrix} \pm 1 & x \\ 0 & 1 \end{pmatrix}$ where $x \in \mathbb{Z}/n\mathbb{Z}$.

2. Consider $D_n$ from the previous exercise. Suppose $a, b \in \mathbb{Z}/n\mathbb{Z}$. Show that $R^a$ and $FR^b$ generate $D_n$ (i.e., $D_n = \langle R^a, FR^b \rangle$) if and only if $a \in (\mathbb{Z}/n\mathbb{Z})^\times$. [Hint: Show that in an arbitrary product of $R^a$-s and $FR^b$-s and their inverses you can collect all the $F$-s on the left side.]

   Recall from class that the matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $\mathrm{SL}(2, \mathbb{Z})$.

3. Find two integers $a, b \in \mathbb{Z}$ such that $A = \begin{pmatrix} 17 & 25 \\ a & b \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and write $A$ explicitly in $\langle S, T \rangle$, i.e., explicitly as a product of integral powers of $S$ and $T$.

4. Show that $\mathrm{SL}(2, \mathbb{Z}) = \langle S, ST \rangle$ and that these two generators $S$ and $ST$ have orders 4 respectively 6. This means that two elements of finite order can generate an infinite group.

5. Show that every homomorphism $f : \mathrm{SL}(2, \mathbb{Z}) \to \mathbb{C}^\times$ has image $\mathrm{Im}\, f \subset \mu_{12} = \{z \in \mathbb{C}^\times \mid z^{12} = 1\}$. [Hint: It's enough to see where the generators go.]

6. Write the permutation
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 7 & 19 & 3 & 1 & 2 & 17 & 20 & 10 & 16 & 8 & 12 & 13 & 11 & 4 & 14 & 5 & 9 & 6 & 15 & 18 \end{pmatrix} \in S_{20}$$
as a product of disjoint cycles.

7. Consider the group homomorphism $f : \mathbb{Z}^3 \to \mathbb{Z}$ given by $f(x, y, z) = 3x + 4y + 5z$. Show that $f$ is surjective and find two vector $u, v \in \mathbb{Z}^3$ which generate $\ker f$.