Math 30810 Honors Algebra 3
Second Midterm

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Do 5.

1. Suppose $n \geq 5$. Show that there are exactly 4 homomorphisms $S_n \to (\mathbb{Z}/2\mathbb{Z})^2$.
2. What is the order of the automorphism group $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})$?
3. Show that (123) and (132) are not conjugate in A_3 or A_4, but are conjugate in S_3.
4. Let G be a group such that $G/Z(G)$ is a cyclic group. Show that G is abelian.
5. (a) Show that $(12\dots n)(i, i+1)(12\dots n)^{-1} = (i+1, i+2)$ for $i+2 \leq n$.
   (b) Show that $(12\dots n)^k(12)(12\dots n)^{-k} = (k+1, k+2)$ for $k+2 \leq n$.
   (c) Deduce that S_n is generated by (12) and $(12\dots n)$.
6. Let $\zeta = e^{2\pi i/3}$, $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$. Let $G = \langle x, y \rangle \subset \mathrm{GL}(2, \mathbb{C})$ be the subgroup generated by x and y.
   (a) Show that x has order 4, y has order 3 and $xy = y^2x$.
   (b) Show that G has order 12 with $G = \{y^b x^a \mid 0 \leq a < 4, 0 \leq b < 3\}$, but $G \not\cong A_4$.
7. Let G be any group. Show that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.
8. Let $H < G$ be groups such that $[G : H]$ is a prime number. Show that if $H < K < G$ is an intermediary subgroup, then $K = H$ or $K = G$. Deduce that if $A_n < G < S_n$ then $G = A_n$ or $G = S_n$.
9. Find all normal subgroups of D_4. (There are 10 subgroups of D_4 of which 6 are normal.)