Math 43900 Problem Solving
Fall 2021
Lecture 13 Brainstorming

Andrei Jorza   Evan O'Dorney

In previous lectures we concentrated on different problem solving topics and techniques, as well as writing up clearly and concisely your solutions. Today we'll concentrate on brainstorming, by which I mean how to poke around effectively when stuck while keeping a clear eye.

1 Problems

1. (Putnam 1975) Suppose n is a sum of two triangular integers, i.e.,

$$n = \frac{a(a+1)}{2} + \frac{b(b+1)}{2}.$$

   Show that $4n+1$ is a sum of two perfect squares and find x, y such that $4n+1 = x^2 + y^2$.

2. (Putnam 1975) For what real numbers b, c do both roots of $z^2 + bz + c = 0$ lie in the interior of the unit circle? Sketch the region in the bc-plane.

3. (Putnam 1983) Let p be an odd prime and $P(X) = 1 + 2X + 3X^2 + \cdots + (p-1)X^{p-2}$. Show that if $a \neq b \in \{0, 1, 2, \ldots, p-1\}$ then $p \nmid P(a) - P(b)$.

4. (Putnam 1976) Find all solutions to the equation $|p^r - q^s| = 1$ where p and q are primes and $r, s \geq 2$.

5. (Putnam 1977) Determine all real numbers x, y, z, w such that

$$x + y + z = w$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}.$$

6. (Putnam 1978) Let $0 < x_i < \pi$ for $i = 1, 2, \ldots, n$. Writing $x = \frac{1}{n}\sum x_i$ show that

$$\prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left(\frac{\sin x}{x}\right)^n.$$

7. (Putnam 1978) Consider n points in the plane. Show that fewer than $2n^{3/2}$ pairs of them are exactly unit distance apart.

8. (Putnam 1968) Suppose $f(x)$ is continuous and $\int_{-\infty}^{\infty} f(x)dx$ exists. Show that

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)dx.$$

9. (Putnam 2010) Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes such that the sum of the numbers in each box is the same? E.g., when $n = 8$ the example $(1, 2, 3, 6)$, $(4, 8)$, $(5, 7)$ shows that the largest k is at least 3.

10. (Putnam 2012) Let $\mathcal{S}$ be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

    (a) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in $\mathcal{S}$;

    (b) If $f(x), g(x)$ are in $\mathcal{S}$ then so are the function $f(x) + g(x)$ and $f(g(x))$;

    (c) If $f(x), g(x)$ are in $\mathcal{S}$ and $f(x) \geq g(x)$ for all $x \geq 0$ then the function $f(x) - g(x)$ is in $\mathcal{S}$.

    Prove that if $f(x), g(x)$ are in $\mathcal{S}$ then so is the function $f(x)g(x)$.

11. (Putnam 2014) Let A be the $n \times n$ matrix whose entry on row i and column j is $1/\min(i, j)$. Compute $\det A$.

12. (Putnam 1960) Consider the sequence $(a_n)_{n \geq 0}$ defined by $a_0 = 0$ and $a_{n+1} = 1 + \sin(a_n - 1)$ for $n \geq 0$. Compute
$$\lim_{n \to \infty} \frac{a_0 + a_1 + \cdots + a_n}{n}.$$

13. (Putnam 1961) The set of pairs of positive reals (x, y) such that $x^y = y^x$ form the straight line $y = x$ and a curve. Find the point at which the curve cuts the line.

14. (Putnam 1963) Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x)^2 - f(y)^2 = f(x + y) f(x - y),$$
for all reals x, y.

15. (Putnam 1967) Find the smallest positive integer n such that we can find a polynomial $nx^2 + ax + b$ with integer coefficients and two distinct roots in the interval $(0, 1)$.