Math 43900 Problem Solving
Fall 2021
Lecture 6 Invariants

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These problems are taken from the textbook, from Engel's *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

The Idea

Often one has to show that a particular configuration is not possible, or that a configuration cannot be obtained from another configuration via certain types of changes. The idea is to attach to a configuration an **invariant** or a **semi-invariant**. The invariant stays the same while the semi-invariant keeps increasing (or decreasing). How do such problems work? To show a configuration is not possible or is not attainable you show that its invariant or semi-invariant is of the wrong type.

Invariants

Easier

1. The numbers from 1 to 1000000 are repeatedly replaced by the sum of their digits until we reach one million single-digit numbers. Which occurs more often: 1 or 2?

2. Show that a 6×6 board cannot be covered with 4×1 pieces. What about a 2006×2006 board? What about an $n \times n$ board?

3. The numbers from 1 to 1000 are arranged in any order on 1000 places numbered 1, 2, ..., 1000. To each integer add its place number. Show that among the 1000 sums there are two with the same last 3 digits (allowing for padding zeros, e.g. the last three digits of 7 are 007).

Harder

4. Each term in a sequence $1, 0, 1, 0, 1, 0, \ldots$ starting with the seventh is the last digit of the sum of the last 6 terms. Prove that the sequence $0, 1, 0, 1, 0, 1$ never occurs.

5. In the following table you may switch the sign of all the numbers of a row, column, or a parallel to one of the diagonals. In particular, you may switch the sign of each corner square. Show that at least -1 will remain in the table.

1	1	1	1
1	1	1	1
1	1	1	1
1	-1	1	1

6. A rectangular floor is covered by 2×2 and 1×4 tiles. One tile got smashed. There is a tile of the other kind available. Show that the floor cannot be covered by rearranging the tiles.

7. The number 99...9 (1997 digits) is written on a board. Each minute, one number written on the board is factored into two factors and erased, each factor is (independently) increased or decreased by 2, and the resulting two numbers are written. Is is possible that at some point all of the numbers on the board are equal to 9?

Semi-invariants

Easier

8. Nine of the unit cells on a 10×10 board are infected. Every minute, the cells with at least 2 infected neighbors become infected. Show that there is always an uninfected cell. [Hint: Look at the perimeter of the infected squares.]

9. Suppose you have real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$. Show that for every permutation $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ of the indices $\{1, 2, \ldots, n\}$ one has

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_n y_{\sigma(n)} \leq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

[Hint: When $i < j$ but $a > b$ what happens when you replace $x_i y_a + x_j y_b$ by $x_i y_b + x_j y_a$?]

Harder

10. Consider the integer lattice in the plane, with one pebble placed at the origin. We play a game in which at each step one pebble is removed from a node of the lattice and two new pebbles are placed at two neighboring nodes, provided that those nodes are unoccupied. Prove that at any time there will be a pebble at distance at most 5 from the origin.

11. Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

Extra problems

Easier

12. If you remove opposite corners of a 10×10 board, is it possible to cover the rest with 49 dominoes (of size 2×1)?

13. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away a stone from the heap, then divide it into two smaller (not necessarily equal) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones by performing the operation finitely many times? [Hint: try to find some expression that stays the same after each move.]

14. Consider the polynomials $P(X) = X^2 + X$ and $Q(X) = X^2 + 2$. Starting with the list $\{P(X), Q(X)\}$. You may keep increasing the list as follows: take any two polynomials f and g in the list (possibly equal), and add to the list $f + g$ or $f - g$ or fg. Is it possible that after finitely many such steps the list contains the polynomial X?

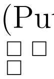
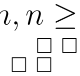
15. A real number is written in each square of an $n \times n$ chessboard. We can perform the operation of changing all signs of the numbers in a row or a column. Prove that by performing this operation a finite number of times we can produce a new table for which the sum of each row and each column is nonnegative.

16. n ones are written on a board. In a step you may erase any two of these numbers, say a and b, and write instead $(a + b)/4$. Repeating this step $n - 1$ times there is only one number left on the board. Show that this number is at least $1/n$. [Hint: Look at the sum of reciprocals of the numbers on the board.]

Harder

17. Start with the number 7^{2016}. At every step you erase the first digit and add it to the remaining number. (E.g., 1234 is replaced by $234 + 1 = 235$.) You stop when you arrive at a 10 digit number. Show that this number has two equal digits. [Hint: think, among other things, of pigeonhole.]

18. You are given an ordered triple of numbers. You are allowed to choose any two of them, say a and b and replace them by $\frac{a+b}{\sqrt{2}}$ and $\frac{a-b}{\sqrt{2}}$. If you start with the triple $(1, \sqrt{2}, 1 + \sqrt{2})$ can you get to the triple $(2, \sqrt{2}, 1/\sqrt{2})$ via a finite number of such changes? [Hint: Play around in the plane first.]

19. (Putnam 2016) Let $m, n \geq 4$. A $(2m - 1) \times (2n - 1)$ board is covered with trominoes  and tetrominoes . What's the smallest number of tiles you need?

20. N men and N women are distributed among the rooms of a mansion. They move among the rooms according to the rules: either

    (a) a man moves from a room with more men than women (counted before he moves) into a room with more women than men, or

    (b) a woman moves from a room with more women than men into a room with more men than women.

    Show that eventually people will stop moving. [Hint: try to find some expression that keeps decreasing after each move.]

Due next week

Write

Please write out clearly and concisely two problems.

Read

In preparation for next class, please look over section on number theory (§5) in the textbook.

Attempt

Please look over the problems from the following lecture. This way you can ask me questions and we can discuss solutions in class.