Math 43900 Problem Solving
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Lecture 9 Combinatorics

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These problems are taken from the textbook, from Engels' *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Combinatorics

Overview

Combinatorics is a rather vast and not particularly well defined subject in problem solving. Roughly speaking it deals with configurations, countings, combinatorial coefficients but also probabilities, graphs and games. When I say configurations I mean it in the most general setting, from sets to integers and from complex numbers to geometry.

While the subject is vast, there are some general patterns and approaches that, while not algorithmic, still can provide you with a starting point, at least to play around.

Basic countings

Here I collect some basing counting results:

1. If you have a set with n elements, there are $n!$ ways to permute the elements. (E.g., abc, bca, cab, bac, cba, and acb.)
2. If you have a set with n elements and wish to choose a subset of k elements, there's $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ways to do this.
3. If you have a set with n elements and wish to choose k elements, with repetition but disregarding order, there are $\binom{n+k-1}{k}$ ways. (The "sticks and stones" counting.)
4. A set with n elements has exactly 2^n subsets, including the empty set. (E.g., abc has $\emptyset$, a, b, c, ab, ac, bc and abc.)

Basic types of questions

1. Is some particular type of configuration possible? Can you show a particular property of a configuration?
   Useful: the pigeonhole principle, induction as well as invariants/semi-invariants.
2. Count a particular number of configurations
   Useful: reduce to known counts, or perhaps simpler counts.

3. Show some properties of combinatorial numbers.
   Useful: lots of strategies, including counts, the binomial formula from calculus, reducing to recursions, generating functions.

A **very** useful tool is the notion of a graph: a **graph** is a collection of vertices and possibly oriented edges connected some vertices to others.

2 Problems

2.1 Configurations

Remember that you've worked on many such problems before, with induction, the pigeonhole principle and invariants.

Easier

1. (Putnam 1983) For a positive integer n let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than 3 times. For example $C(8) = 5$. Find a polynomial $P(n)$ such that $C(n) = \lfloor P(n) \rfloor$.

Harder

2. (Putnam 1980) Let $A_1, A_2, \ldots, A_{1066}$ be distinct subsets of a finite set X such that $|A_i| > |X|/2$ for all i. Prove that there exist ten elements $x_1, \ldots, x_{10}$ in X such that each A_i contains at least one of $x_1, \ldots, x_{10}$.

2.2 Combinatorial coefficients

The most useful formula for binomial coefficients is the binomial expansion formula:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

In calculus you also learn that this works with Taylor series. If $\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-(n-1))}{n!}$ then

$$(x+1)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n$$

Easier

3. Show that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

Harder

4. (Putnam 1983) Let k be a positive integer and $m = 6k-1$. Show that $\sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}$ is never 0.

2.3 Combinatorics and probabilities

By far the most useful formula for probabilities (outside of calculus) is that expectated value is additive: if X and Y are random variables then

$$E[X+Y] = E[X] + E[Y].$$

You can think of expected value as "mean" or "average". (Variance is also additive but only for *uncorrelated* variables.)

Easier

5. Let v, w be distinct, randomly chosen roots of the equation $z^{2017} - 1 = 0$. Find the probability that $\sqrt{2+\sqrt{3}} \leq |v+w|$.

Harder

6. (Putnam 1974) An unbiased coin is tossed n times. What is expected value of $|H - T|$ where H is the number of heads and T is the number of tails?

2.4 Extra problems

Easier

7. (a) Show that you can cover a 9×5 board with  trominoes.

   (b) Show that you can cover an $m \times n$ board with  trominoes if and only if $3 \mid mn$ and, if one of the dimensions is 3, then the other is even.

8. Let A and B be two sets. Find all sets X such that $A \cap X = B \cap X = A \cap B$ and $A \cup B \cup X = A \cup B$.

9. For two sets A and B define $A \Delta B := (A - B) \cup (B - A)$. Fix a set A and consider the function $f(X) := X \Delta A$ of a variable X which represents a set. Show that $f \circ f = \mathrm{id}$ and conclude that f is injective.

10. In the coordinate plane draw the integer grid, with vertical and horizontal lines at integer intercepts. A path in the plane is said to be **good** if (a) it only goes along the integer grid lines and (b) if it only goes up or to the right. Show that the number of good paths from $(0,0)$ to (m,n) is exactly $\binom{m+n}{m}$.

11. You draw n great circles on a sphere. (A great circle on a sphere is one whose center is the same as the center of the sphere.) No three of these n circles intersect in one point. In how many regions do the n circle divide the surface of the sphere?

12. A subset S of $\{1, 2, \ldots, n\}$ is said to be connected if the following condition holds: if $a \in S$ then either $a+1$ or $a-1$ is also in S.

    (a) Find the number of connected subsets when $n = 7$.

    (b) Harder: Find the number of connected subsets in general.

13. Show that $\binom{n}{k} = \binom{n}{n-k}$.

14. Compute the Taylor expansion of $\sqrt{1-4x}$ around $x = 0$. (This is useful to compute the Catalan numbers.)

15. Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. Also show that $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

16. Show that $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$. Deduce that if k and n are coprime then $k \mid \binom{n-1}{k-1}$.

17. For each permutation $a_1, \dots, a_{10}$ of the integers $1, 2, \dots, 10$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_9 - a_{10}|.$$

Find the average value of all such sums.

Harder

18. Recall the notion of a good path from Exercise 10. A good path is called super-good if it never goes above the diagonal line $y = x$.
    (a) Show that the number of good paths from $(0, 0)$ to (n, n) that aren't super-good is the same as the number of good paths from $(-1, 1)$ to (n, n). [Hint: Can you come up with a procedure that transforms a non-super-good path into a super-good path from $(-1, 1)$ to (n, n)?]
    (b) How many super-good paths are there?

19. Let $c(n)$ be the number of ways to partition an $(n+2)$-gon into triangles using only diagonals. E.g., $c(1) = 1$, $c(2) = 2$. Show that

$$c(n) = c(0)c(n-1) + c(1)c(n-2) + \cdots + c(n-1)c(0)$$

and compute $c(3), c(4), c(5)$.

20. Show that $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

21. A marginally harder variant of the previous problem: show that

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}.$$

22. Show that $\sum_{0 \le k \le n/2} \binom{n-k}{k} = F_n$, the Fibonacci number where $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$.

23. (Putnam 2016) Given a positive integer n, let $M(n)$ be the largest integer m such that $\binom{m}{n-1} > \binom{m-1}{n}$. Compute $\lim_{n\to\infty} \frac{M(n)}{n}$.

24. (Putnam 2016) Let A be a $2n \times 2n$ matrix with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability 1/2. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

25. An exam consists of 3 problems selected randomly from a list of $2n$ problems, where n is an integer greater than 1. For a student to pass, he needs to solve correctly at least two of the three problems. Knowing that a certain student knows how to solve exactly half of the $2n$ problems, find the probability that the student will pass the exam.