Math 40520 Theory of Number
Homework 7

Due Wednesday, 11/9

Do 5.

1. Suppose $3^{10} \mid \binom{3n}{n}$ for a positive integer n. Show that $n > 3^9$.

2. In this exercise you will use the model $\pi(x) \approx \int_2^x \frac{dt}{\ln t}$ with $d\pi(x) \approx \frac{dx}{\ln x}$. Show that
$$\sum_{p \le n} \frac{1}{p} \approx \ln \ln n.$$

3. In this exercise you will use the model $\pi(x) \approx \int_2^x \frac{dt}{\ln t}$ with $d\pi(x) \approx \frac{dx}{\ln x}$. Show that
$$\prod_{p \le n} p \approx e^n,$$
and therefore our construction of n consecutive integers which are not square-free will yield an answer at most approximately e^{2n}.

4. What is $v_3((2^n - 1)(2^n - 2)(2^n - 2^2) \cdots (2^n - 2^{n-1}))$ when $n = 2022$?

5. Determine $v_3(5^n - 1)$ as a function of $v_3(n)$. (Careful, the answer is a piece-wise defined function.)

6. What is $v_{11}(3^{146410} - 2^{146410})$?

7. What is $v_7(3^{5402250} - 2^{31513125})$? [Hint: This is not as hard as it looks. Rewrite the difference as a sum of two expressions of the form $a^n - b^n$.]

8. Let a be a positive integer. Find the smallest positive integer k such that $2^{2022} \mid 2049^k - 1$.

9. Suppose x and $2 \le k \le n$ are integers. Show that
$$v_p\left(\binom{n}{k} x^k\right) - v_p(n) - v_p(x) = (k-1)v_p(x) - \frac{k - s_p(k)}{p-1} + v_p((n-1)(n-2)\cdots(n-k+1))$$
and therefore that $v_p\left(\binom{n}{k} x^k\right) > v_p(n) + v_p(x)$ whenever $v_p(x) > \frac{1}{p-1}$. Is this enough to prove LTE?