Math 40520 Theory of Number
Homework 8

Due Wednesday 11/16

Do 5.

1. What is the order of 5 modulo 2^{100}?
2. Solve $x^3 \equiv 8 \pmod{7^5}$. You may use that 3 is a primitive root mod 7^5, for instance by using one of the questions on this set. [Hint: Use primitive roots, just like we did with such equations mod primes.]
3. Suppose $p > 2$ is a prime and a is an integer coprime to p. Show that a is a perfect square modulo p^n if and only if
$$a^{\varphi(p^n)/2} \equiv 1 \pmod{p^n}.$$
[Hint: Almost identical to the Legendre symbol case.]
4. Let p be an odd prime. Suppose that $a \neq 0$ is a square mod p. Show that a is a square mod p^n for every $n \geq 1$.
5. Suppose $p > 2$ is a prime, $n \geq 2$ and a is an integer such that a is a primitive root modulo p^2. Show that a is then also a primitive root modulo p^n for all n. [Hint: Similar to the proof from class, but using a, not the specially constructed number.]
6. Write $f(x) = [3, 1, 4, 1, 5, 9, 2, 6, x]$ as a function of x in the form of a linear fractional transformation.
7. Suppose $x > 0$ is a real number. Find positive integers $a_0, \ldots, a_n$ such that $\dfrac{7x+3}{5x+2} = [a_0; a_1, \ldots, a_d, x]$.
8. Determine the complete continued fraction expansion of $\frac{1982}{2022}$.
9. Find a fraction $\frac{a}{b}$ with $0 < a, b < 100$ with the longest possible continued fraction expansion among all such fractions.