Math 40520 Theory of Number
Homework 9

Due Wednesday 11/30

Do 5.

1. Textbook exercise 5.3 on page 121.
2. Textbook exercise 5.4 on page 121.
3. A fraction $\frac{a}{b} \approx 0.515287517$ has a and b with 4 digits. What are they?
4. Continued fractions of the form $[a_0; a_1, a_2, \ldots]$ make sense for real numbers $a_n > 0$. Show that they converge whenever $\sum a_n$ diverges (see Theorem 5.3.8 in the textbook).
5. (IMO 2019) Determine all positive integers m, n with the property that
$$m! = (2^n - 1)(2^n - 2)(2^n - 2^2) \cdots (2^n - 2^{n-1}).$$
[Hint: Combine the previous two questions about this equation.]
6. Suppose a real number x has the following continued fractions: $x = [4; 3, 2, 5, 1, a_5, a_6, a_7, \ldots]$ where a_n are integers. Show that
$$\left| x - \frac{193}{45} \right| < \frac{1}{3735}.$$
7. Find a polynomial $P(X) \in \mathbb{Q}[X]$, which is a product of 2 linear and 2 quadratic irreducible polynomials in $\mathbb{Q}[X]$, and which has the the following 6-decimal approximation:
$$P(X) \approx X^6 - 5.380846X^5 + 1.723134X^4 + 21.724378X^3 - 14.273383X^2 - 10.412687X + 1.997512.$$
8. Suppose $P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0 \in \mathbb{Z}[X]$ has a rational root $\frac{a}{b}$, written in lowest terms. Show that $a \mid a_0$ and $b \mid a_n$.
9. In the plane consider an equilateral triangle ABC where A is at the origin, B is on the x-axis, and C is above the x-axis. I pick a point D, whose coordinates are positive integers, and whose x-coordinate is ≤ 10864, such that the rays AC and AD are as close as possible. What are the coordinates of D?