Math 43900 Problem Solving
Fall 2022
Lecture 14 Which problems do I choose?

Andrei Jorza

Putnam 1964

A1 Let $A_1, A_2, A_3, A_4, A_5, A_6$ be distinct points in the plane. Let D be the longest distance between any pair, and d the shortest distance. Show that $\frac{D}{d} \geq \sqrt{3}$.

A2 α is a real number. Find all continuous real-valued functions $f : [0, 1] \to (0, \infty)$ such that $\int_0^1 f(x)dx = 1, \int_0^1 xf(x)dx = \alpha, \int_0^1 x^2 f(x)dx = \alpha^2$.

A3 Let $P_1, P_2, \dots$ be a sequence of distinct points which is dense in $(0, 1)$. The points $P_1, \dots, P_{n-1}$ decompose the interval into n parts and P_n decomposes one of these into two parts. Let a_n, b_n be the lengths of these two intervals. Prove that $\sum_{n=1}^{\infty} a_n b_n (a_n + b_n) = \frac{1}{3}$.

A4 The sequence of integers (u_n) is bounded and satisfies $u_n = \frac{u_{n-1} + u_{n-2} + u_{n-3}u_{n-4}}{u_{n-1}u_{n-2} + u_{n-3} + u_{n-4}}$. Show that it is periodic for sufficiently large n.

A5 Find a constant k such that for any positive reals (a_n),

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + a_2 + \cdots + a_n} \leq k \sum_{n=1}^{\infty} \frac{1}{a_n}.$$

A6 S is a finite set of collinear points. Let k be the maximum distance between any two points of S. Given a pair of points of S a distance $d < k$ apart, we can find another pair of points of S also a distance d apart. Prove that if two pairs of points of S are distances a and b apart, then a/b is rational.

B1 Let (a_n) be positive integers such that $\sum \frac{1}{a_n} < \infty$. Let b_n be the number of a_m which are $\leq n$. Prove $\lim b_n/n = 0$.

B2 S is a finite set. Let $A_1, \dots, A_k$ be distinct subsets of S such that any two of them meet. Assume no other subset of S meets all the A_i. Show that $k = 2^{n-1}$.

B3 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for every $\alpha > 0$, $\lim_{n\to\infty} f(n\alpha) = 0$. Prove that $\lim_{x\to\infty} f(x) = 0$.

B4 n great circles on the sphere are in general position (in other words at most two circles pass through any two points on the sphere). How many regions do they divide the sphere into?

B5 Let (a_n) be a strictly monotonic increasing sequence of positive integers. Let b_n be the least common multiple of $a_1, a_2, \dots, a_n$. Prove that $\sum 1/b_n$ converges.

B6 Show that the unit disc in the plane cannot be partitioned into two disjoint congruent subsets.