# Math 43900 Problem Solving
# Fall 2022
# Lecture 8 Sequences and series

Andrei Jorza

These problems are taken from the textbook, from Engel's *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

## 1 Sequences and series

### Overview

Calculus is certainly a vast topic in problem solving and you'll have to use pretty much everything you know from calculus to solve typical problems. Alas there's no way around algebraic manipulations and often an easy problem seems impossibly unless you see some weird algebraic manipulation. Nevertheless there are some tricks and some useful facts that can help you approach problems.

Typically problems with sequences involve either

1. finding the general term of a sequence of proving something about the general term of a sequence or

2. computing the limit of a sequence

A related topic is that of infinite series and infinite products where again I can identify two things:

3. showing that some series or products converge and

4. computing the values of some converging series or product.

Typically when asked to compute the value of a series or a product the actual computation might be easier than showing that the series converges and what you are doing is actually sensible. This is often true when having to compute integrals (a later topic) and you use Taylor expansions to compute the value, but showing convergence is harder.

### Basic results

Remember all the convergence tests from calculus: bounded monotonous sequence test, ratio test and harmonic series test for series, telescoping sums and products, etc. In addition, the following are useful:

1. Cauchy's criterion: a sequence $(x_n)$ converges if and only if as $m$ and $n$ grow, $|x_m - x_n|$ becomes small.
2. Stirling's approximation: $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} e^{\frac{\theta_n}{12n}}$ for some $\theta_n \in (0,1)$. (See textbook section 3.2.11.)
3. Recall that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.
4. You can go between infinite products and infinite series using logarithms and exponentiations.

# 2 Problems

## 2.1 Sequences and their limits

**Easier**

1. Compute the limit $\sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}$.

**Harder**

2. Suppose $k \in \mathbb{Z}_{\geq 1}$ and $x \in \mathbb{R}$. Show that

$$\lim_{n\to\infty} \binom{n}{k} \left(\frac{x}{n}\right)^k \left(1 - \frac{x}{n}\right)^{n-k} = \frac{x^k}{e^x k!}$$

## 2.2 Series and products

**Easier**

3. (Putnam 1984) Compute the sum $\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$.
4. (Putnam 1982) For $x > 0$ let $B_n(x) = 1^x + 2^x + \cdots + n^x$. Prove or disprove the convergence of $\sum_{n=2}^{\infty} \frac{B_n(\log_n 2)}{(n \log_2(n))^2}$.
5. (Putnam 1981) Compute $\lim_{n\to\infty} \frac{1}{n^5} \sum_{a,b=1}^{n} (5a^4 - 18a^2b^2 + 5b^4)$.
6. (Putnam 1977) Compute $\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$.
7. (Putnam 1976) Compute $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left( \left\lfloor \frac{2n}{k} \right\rfloor - 2 \left\lfloor \frac{n}{k} \right\rfloor \right)$.

**Harder**

8. (Putnam 1978) Compute $\sum_{m,n=1}^{\infty} \frac{1}{m^2n + mn^2 + 2mn}$ as a rational number.

9. (Putnam 1975) Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n \geq 0$. Show that $\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e$.

10. (Putnam 1977) For $0 < x < 1$ compute $\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$.

## 2.3 Extra exercises

**Easier**

11. Compute $\lim \sqrt[n]{n}$.

12. Suppose $a \in \mathbb{Z}_{\geq 1}$ is such that $a \equiv 3 \pmod 4$.

    (a) Let $S_n = 1 - \binom{n}{2}a + \binom{n}{4}a^2 - \binom{n}{6}a^3 + \cdots$. Show that $2S_n = (1 + i\sqrt{a})^n + (1 - i\sqrt{a})^n$.

    (b) Find a recurrence relation for $S_n$ and show by induction that $2^{n-1} \mid S_n$.

13. In base $b$ write the number

$$x_n = \underbrace{11\ldots1}_{n-1}\underbrace{22\ldots2}_{n}5$$

Suppose that for $n$ large enough, $x_n$ is a perfect square.

    (a) Show that $b - 1$ is a perfect square. [Hint: Look at $b\sqrt{x_n} - \sqrt{x_{n+1}}$.]

    (b) (Harder) Show that $b = 10$.

14. Compute (and show the limit exists) the continued fraction

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}}$$

15. (Putnam 1993) Let $x_0, x_1, x_2, \ldots$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n+1}x_{n-1} = 1$ for all $n \geq 1$. Show that there exists a real $a$ such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

16. Does
$$\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right)$$
converge?

17. Does $\sum \sin(\pi\sqrt{n^2+1})$ converge?

18. Compute
$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{n-1}+\sqrt{n}}$$

19. Compute $(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$.

20. (Putnam 1986) Evaluate the sum $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2+n+1)$.

**Harder**

21. (Putnam 1990) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n}-\sqrt[3]{m}$ for nonnegative $m, n$?

22. Show that $\sqrt{1+\sqrt{2+\sqrt{3+\cdots}}}$ converges.

23. Let $a_0=0, a_1=1, a_2=2, a_3=6$ and
$$a_{n+4}=2a_{n+3}+a_{n+2}-2a_{n+1}-a_n$$
Show that $n \mid a_n$ for $n \geq 1$.

24. Compute
$$\lim_{n\to\infty} n^2 \int_0^{1/n} x^{x+1}dx$$

25. (Putnam 2016) Let $x_0, x_1, \ldots$ be the sequence such that $x_0=1$ and for $n\geq 0$, $x_{n+1}=\ln(e^{x_n}-x_n)$. Show that $\sum x_n$ converges and find its sum.

26. Suppose $x_n$ are real such that $x_{n+1}\leq x_n+\frac{1}{n^2}$. Show that $\lim x_n$ exists.

27. Compute the product
$$(1-4/1)(1-4/9)(1-4/25)\cdots$$