Math 43900 Problem Solving
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Lecture 10 Matrices

Andrei Jorza

These problems are taken from the textbook, from Engel's *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Matrices

Overview

The way matrices show up in problem solving problems involves the following three main themes:

1. algebraic manipulations of matrices (they can be multiplied and the operation is not commutative),
2. determinants and eigenvalues of matrices,
3. matrices as defining linear maps on vector spaces.

Basic results

1. You can always add two $m \times n$ matrices.
2. You can always multiply an $m \times n$ matrix and an $n \times p$ matrix to get an $m \times p$ matrix.
3. The **trace** of a matrix $\operatorname{Tr} A$ is the sum of its diagonal terms. It has the property that $\operatorname{Tr}(A+B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$ and $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for all matrices A and B.
4. The **determinant** of a matrix $\det A$ is a polynomial expression in the entries of the matrix A and satisfies the following properties:
   (a) The determinant of (a_{ij}) is $\sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$, where S_n is the group of permutations and $\varepsilon(\sigma)$ is the sign. The sign ε is multiplicative and if τ is a k-cycle then $\varepsilon(\tau) = (-1)^{k-1}$.
   (b) If in a matrix $A = (a_{ij})$ you write $A_{p,q}$ for the $(n-1) \times (n-1)$ where you eliminate the p-th row and q-th column from A then
   $$\det(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n-1} a_{1,n} \det A_{1,n}$$
   (c) A is invertible if and only if $\det A \neq 0$.
   (d) $\det(AB) = \det(A)\det(B)$ for all matrices A and B.
   (e) If you swap two rows or columns of a matrix A to obtain a matrix B then $\det(B) = -\det(A)$.
   (f) If in a matrix A you add a multiple of one row to a different row to get a matrix B then $\det(B) = \det(A)$. The same is true if you add a multiple of a column to a different column.

5. Suppose A is an $n \times n$ matrix. If you can find a *nonzero* vector v (i.e., an $n \times 1$ matrix consisting of a single column) and a scalar α such that $Av = \alpha v$ then α is said to be an **eigenvalue** of A with **eigenvector** v.

6. If A is an $n \times n$ matrix the **characteristic polynomial** of A is the monic degree n polynomial

$$P_A(X) = \det(XI_n - A)$$

   (a) A scalar α is an eigenvalue of A if and only if it is a root of $P_A(X)$. The roots of $P_A(X)$ are **the** eigenvalues of A and are counted with multiplicity if they are not distinct. E.g., I_n has n eigenvalues all equal to 1.

   (b) $P_A(X) = X^n - (\operatorname{Tr} A)X^{n-1} + \cdots + (-1)^n \det(A)$.

   (c) Since we know the relation between the coefficients of a polynomial and its roots we deduce that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A then

$$\begin{aligned} \operatorname{Tr}(A) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\ \det(A) &= \lambda_1 \lambda_2 \cdots \lambda_n \end{aligned}$$

   (d) The Cayley-Hamilton theorem: If you plug A into the polynomial $P_A(X)$ you always get the 0 matrix, $P_A(A) = O$.

   (e) If A and B are matrices then $P_{AB}(X) = P_{BA}(X)$ as polynomials.

7. A big result in linear algebra says that for any matrix A (over $\mathbb{C}$) you can find an invertible matrix S such that the conjugate SAS^{-1} has a very special shape: the **Jordan canonical form**. In fact the Jordan canonical form SAS^{-1} has the n eigenvalues on the diagonal but much more is true: SAS^{-1} is block diagonal and each block is of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \ldots \\ 0 & \lambda & 1 & \ldots \\ & & \ddots & \ddots \\ 0 & \ldots & 0 & \lambda \end{pmatrix}$$

with an eigenvalue λ on the diagonal and 1-s off diagonal. E.g., for a 2×2 matrix the possible Jordan canonical forms are

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ for } \lambda_1 \neq \lambda_2 \text{ and } \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ or } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

8. (**VERY USEFUL**) Suppose A is an $n \times n$ matrix and $Q(X)$ is any polynomial. If the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $Q(A)$ (also an $n \times n$ matrix) are $Q(\lambda_1), \ldots, Q(\lambda_n)$.

2 Problems

2.1 Determinants, traces, characteristic polynomials and eigenvalues

Easier

1. (Putnam 1978) Let $a \neq b$ and $p_1, \ldots, p_n$ be real numbers, and let $F(X) = (p_1 - X) \cdots (p_n - X)$. Let M be the $n \times n$ matrix which has $p_1, \ldots, p_n$ on the diagonal, a above the diagonal, and b below the diagonal. Show that

$$\det M = \frac{bF(a) - aF(b)}{b - a}.$$

2. (Putnam 1969) Show that $\det(|i - j|)_{1 \leq i,j \leq n} = (-1)^{n-1}(n-1)2^{n-2}$.

3. Let D_n be the $(n-1)\times(n-1)$ determinant that has $3, 4, \ldots, n+1$ on the diagonal and 1's everywhere else. Show that $\{D_n/n!\}$ is unbounded.

4. (Putnam 2018) Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n-1)\times(2^n-1)$ matrix whose (i,j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset; \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

Harder

5. (Putnam 1984) Let $M(x) = (m_{i,j})$ be the $2n \times 2n$ matrix with entries $m_{i,j} = x$ if $i = j$, $m_{i,j} = a$ if $i \neq j$ and $i + j$ is even, and $m_{i,j} = b$ if $i \neq j$ and $i + j$ is odd. Compute $\lim_{x\to a} \dfrac{\det M(x)}{(x-a)^{2n-2}}$.

6. (Putnam 1985) Let $G = \{M_1, \ldots, M_r\}$ be a finite set of $n \times n$ matrices which form a group under matrix multiplication. Suppose $\sum_{i=1}^{r} \mathrm{Tr}(M_i) = 0$. Show that $\sum_{i=1}^{r} M_i = 0_{n\times n}$.

2.2 Algebraic operations and linear algebra

Easier

7. Compute $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n$ and $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n$ for all n.

8. Suppose $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ is a converging power series. Show that $f(SAS^{-1}) = Sf(A)S^{-1}$.

Harder

9. (Putnam 2015) Let S be the set of all 2×2 real matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

10. (Putnam 1986) Let A, B, C, D be $n \times n$ matrices with complex entries such that: AB^t and CD^t are symmetric and $AD^t - BC^t = I_n$. Show that $A^tD - C^tB = I_n$.

11. (Putnam 1987) Let M be a $2n \times n$ matrix with complex entries such that whenever $(z_1, \ldots, z_{2n})M = O_{1\times n}$ with complex z_i, not all 0, then at least one z_i is not real. Show that for any real $r_1, \ldots, r_{2n}$ there exist complex $z_1, \ldots, z_n$ such that $\mathrm{Re}(M(z_1, \ldots, z_n)^t) = (r_1, \ldots, r_{2n})^t$.

2.3 Extra problems

Easier

12. Show that you can never find two $n\times n$ matrices A and B with real coefficients such that $AB - BA = I_n$.

13. Consider an $n \times (n+1)$ matrix $A = (a_{ij})$. For a column k write A_k for the $n \times n$ matrix you obtain from A by removing the k-th column. Show that

$$a_{11} \det A_1 - a_{12} \det A_2 + \cdots + (-1)^{n+1} a_{1,n+1} \det A_{n+1} = 0$$

14. Suppose $P(X)$ is a polynomial and A is an $n \times n$ matrix such that $P(A) = 0$. Show that the eigenvalues of A are among the roots of $P(X)$.

15. This is an application of Exercise 21. Suppose X is an antisymmetric matrix, i.e., of the form $X = -X^t$. (Think $\begin{pmatrix} & x \\ -x & \end{pmatrix}$.) Show that every eigenvalue of X is of the form ai where $i = \sqrt{-1}$ and $a \in \mathbb{R}$.

16. Show that $A^k = 0$ for some $k \geq 0$ if and only if all the eigenvalues of A are 0 in which case $A^n = 0$ as well.

17. (Putnam 1994) Let A and B be 2 by 2 matrices with integer entries such that $A, A+B, A+2B, A+3B$ and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

18. Let $p < m$ be positive integers. Show that $\det(\binom{m+i}{j})_{i,j \leq p} = 1$.

19. Suppose (x_n) is a sequence defined by the linear recurrence $x_{n+2} = ax_{n+1} + bx_n$ for all $n \geq 0$. Show that
$$\begin{pmatrix} x_{n+2} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}$$
and conclude that for $n \geq 1$, x_n is the first entry of the matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$.

20. A useful application of Exercise 7. Show that if $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is an absolutely convergent power series then $f\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\right) = \begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}$ and $f\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = \begin{pmatrix} f(\lambda) & f'(\lambda) \\ 0 & f(\lambda) \end{pmatrix}$.

21. If u and v are $n \times 1$ column matrices write $\langle u, v \rangle = u^t v$ for the dot product of the two vectors. If A is an $n \times n$ matrix show that $\langle u, Av \rangle = \langle A^t u, v \rangle$. Show that $\langle v, \overline{v} \rangle \geq 0$, where $\overline{v}$ is the complex conjugate of v.

22. If $A = (a_{ij})$ show that $\mathrm{Tr}(A \cdot A^t) = \sum_{i,j} a_{ij}^2$.

Harder

23. Suppose A is an $n \times n$ real matrix such that $A^2 = A + I_n$. Show that $\det(A) < 2^n$. In fact show that $\det(A) \leq \left(\frac{1+\sqrt{5}}{2}\right)^n$.

24. Suppose X is a real matrix with $X + X^t = I_n$. Show that $\det X \geq \frac{1}{2^n}$.

25. Compute the determinant of the matrix (a_{ij}) where $a_{ii} = 2$ and if $i \neq j$ then $a_{ij} = (-1)^{i-j}$.

26. Let A and B be 3×3 matrices with real entries such that $\det A = \det B = \det(A+B) = \det(A-B) = 0$. Show that $\det(xA + yB) = 0$ for all real numbers x, y.

27. Let n be an odd positive integer. Suppose A is an $n \times n$ matrix whose square A^2 is either 0 or I_n. Show that $\det(A + I_n) \geq \det(A - I_n)$.

28. Suppose A and B are commuting $n \times n$ matrices with real entries such that $\det(A + B) \geq 0$. Show that $\det(A^k + B^k) \geq 0$ for all $k \geq 1$.

29. (Putnam 1996) Show that there exists no complex matrix A such that $\sin(A) = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}$.

30. Suppose A and B are $n \times n$ real matrices such that $\mathrm{Tr}(A \cdot A^t + B \cdot B^t) = \mathrm{Tr}(A \cdot B + A^t \cdot B^t)$. Show that $A = B^t$.