Math 43900 Problem Solving
Fall 2023
Lecture 11 Inequalities

Andrei Jorza

These problems are taken from the textbook, from Engel's *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Basics

Inequalities are a frequent and difficult topic on math competitions, and they are at the core of a huge number of results in analysis. Problem solving inequalities tend to be on the tricky side with ingenious algebra necessary to reduce them to some known inequalities. Nevertheless a handful of basic examples can be helpful in proving a large number of inequalities.

The basic inequalities:

1. By far the most useful inequality is that $x^2 \geq 0$ for all x real.
2. **AM-GM:** If $x_1, \ldots, x_n \geq 0$ then
$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$
with equality when $x_1 = x_2 = \ldots = x_n$.
3. **Cauchy-Schwarz:** If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are real numbers then
$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \geq (x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2$$
with equality when $x_1 = \lambda y_1$, $x_2 = \lambda y_2$, $\ldots$, $x_n = \lambda y_n$ for a scalar λ.
4. **Chebyshev's inequality**: If $x_1 \leq x_2 \leq \ldots \leq x_n$ and $y_1 \leq y_2 \leq \ldots \leq y_n$ then
$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \geq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_n y_{\sigma(n)} \geq x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1$$
for any permutation σ. The idea of the proof is that in a sum of the form $\sum a_i b_i$ if you interchange b_i and b_{i+1} then sum grows if and only if $(a_i - a_{i+1})(b_i - b_{i+1}) < 0$.

Needless to say you may use any method from calculus to show inequalities, from minimization/maximization to Lagrange multipliers. Typically, however, reducing inequalities to the basic ones via algebraic manipulations is the most effective strategy. Brute force methods sometimes work, but they are very laborious.

Inequalities come is lots of guises but the following are major themes in problem solving:

1. Inequalities based on AM-GM
2. Inequalities based on Cauchy-Schwarz
3. Inequalities in geometry, where a useful fact is the triangle inequality.
4. Inequalities in calculus

2 Problems

2.1 AM-GM, Completing the square, Cauchy-Schwarz, Chebyshev

Easier

1. (Putnam 1985) Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

2. Find the maximum of the function $f(x, y, z) = 5x - 6y + 7z$ on the ellipsoid $2x^2 + 3y^2 + 4z^2 \leq 1$.

3. If $a_1 + a_2 + \cdots + a_n = n$, show that $a_1^4 + a_2^4 + \cdots + a_n^4 \geq n$.

Harder

4. (Putnam 1977) Suppose $a_1, \ldots, a_n$ are real numbers and A is a real number such that

$$A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n-1} \left(\sum_{i=1}^{n} a_i \right)^2.$$

   Show that $A < 2a_i a_j$ for all $i \neq j$.

5. (Putnam 1996) Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$ find, with proof, the largest possible value of

$$x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1.$$

2.2 Inequalities in calculus and geometry

Easier

6. (Putnam 1966) Let a, b, c be the lengths of the three sides of a triangle, let $s = (a+b+c)/2$, and r be the radius of the inscribed circle. Show that

$$\frac{1}{(s-a)^2} + \frac{1}{(s-b)^2} + \frac{1}{(s-c)^2} \geq \frac{1}{r^2}.$$

7. (Putnam 1967) Let $f(x) = \sum_{k=1}^{n} a_k \sin(kx)$ where $a_1, \ldots, a_n$ are real numbers. Given that $|f(x)| \leq |\sin(x)|$ for all x show that $|a_1 + 2a_2 + \cdots + na_n| \leq 1$.

8. (Putnam 1988) Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$ then $y(y-1) \leq x^2$.

Harder

9. (Putnam 1967) Let $f(x, y)$ be a real-valued function on the unit disc $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Suppose $|f(x, y)| \leq 1$ on D and f has partial derivatives everywhere in D. Show that at some point (x_0, y_0) in the interior of D, $|f_x(x_0, y_0)|^2 + |f_y(x_0, y_0)|^2 \leq 16$.

10. (Putnam 1972) Let $n_1 < n_2 < \ldots < n_k$ be positive integers. Show that $P(z) = 1 + z^{n_1} + \cdots + z^{n_k}$ has no roots in the circle $|z| < \frac{\sqrt{5}-1}{2}$.

2.3 Miscellaneous

Easier

11. (VTRMC 2017) Determine the number of real solutions to the equation $\sqrt{2 - x^2} = \sqrt[3]{3 - x^3}$.

12. (Putnam 1996) Show that for all positive integers n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Harder

13. (Putnam 1971) Let $\delta(m)$ be the largest odd divisor of m. Show that for all integers $n \geq 1$:

$$\left| \sum_{k=1}^{n} \frac{\delta(k)}{k} - \frac{2n}{3} \right| \leq 1.$$

2.4 Extra problems

Easier

14. Show that for all real numbers x,

$$2^x + 3^x - 4^x + 6^x - 9^x \leq 1.$$

15. Show that $x^4 + 4x + 3 \geq 0$ for all real x. Find all positive integers n such that the equation

$$nx^4 + 4x + 3 = 0$$

has a real root.

16. Show that the positive real numbers $a_0, a_1, \ldots, a_n$ form a geometric progression if and only if

$$(a_0a_1 + a_1a_2 + \cdots + a_{n-1}a_n)^2 = (a_0^2 + a_1^2 + \cdots + a_{n-1}^2)(a_1^2 + a_2^2 + \cdots + a_n^2).$$

17. Suppose $f, g : [0,1] \to \mathbb{R}$ are continuous functions. Show that

$$\int_0^1 f(x)^2 dx \int_0^1 g(x)^2 dx \geq \left(\int_0^1 f(x)g(x)dx \right)^2.$$

18. (VTRMC 2017) Find all nonnegative integers m and n such that $m^2 + 2 \cdot 3^n = m(2^{n+1} - 1)$.

Harder

19. Suppose $x_1, \ldots, x_n \in (1/4, 1)$. Show that

$$\log_{x_1}(x_2 - 1/4) + \log_{x_2}(x_3 - 1/4) + \cdots + \log_{x_n}(x_1 - 1/4) \geq 2n.$$

20. Suppose $a_1, \ldots, a_n$ are real numbers such that $a_1 + \cdots + a_n \geq n^2$ and $a_1^2 + \cdots + a_n^2 \leq n^3 + 1$. Show that $a_1, \ldots, a_n \in [n-1, n+1]$.

21. Consider real numbers $x_0 > x_1 > x_2 > \cdots > x_n$. Show that

$$x_0 + \frac{1}{x_0 - x_1} + \frac{1}{x_1 - x_2} + \cdots + \frac{1}{x_{n-1} - x_n} \geq x_n + 2n.$$

22. Show that if $0 < a, b < \pi/2$ then

$$\frac{\sin^3 a}{\sin b} + \frac{\cos^3 a}{\cos b} \geq \sec(a - b).$$

23. Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

24. Show that in a triangle with sides a, b, c and area A one has

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}A.$$

25. (Romanian National 1999) Find all positive real x, y such that

$$\begin{cases} 4^{-x} + 27^{-y} = \frac{5}{6} \\ 27^y - 4^x \leq 1 \\ \log_{27} y - \log_4 x \geq \frac{1}{6}. \end{cases}$$