Math 43900 Problem Solving
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Lecture 12: Functional equations

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These problems are taken from the textbook, from Engel's *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Functions and functional equations

In physics and calculus, you've seen differential equations where you were supposed to determine a particular function $f(x)$ satisfying a particular equation involving differentials. These are special examples of "functional equations", i.e., problems where you were supposed to determine a particular function $f(x)$ given only an equation satisfied by $f(x)$. They are a popular topic in math contests and solving them requires ingenuity and playfulness.

Example 1 (Cauchy's functional equation)**.** The most classical example of a simple (nondifferential) functional equation is to determine functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f(x + y) = f(x) + f(y)$$

As it stands the example has countless solutions (and I mean it in a technical way, there are uncountably many solutions). However, assuming mild properties of $f(x)$ one can show that $f(x) = ax$ for a fixed $a \in \mathbb{R}$ are the only solutions. This is the case when $f(x)$ is assumed to be continuous, or even integrable.

Remark 1. A large number of functional equations can be reduced to Cauchy's functional equation via alegbraic manipulations.

I identified 3 main topics:

1. Functional equations with integers, where you use the fact that the integers are discrete.
2. Functional equations over $\mathbb{R}$ where you use algebraic manipulations.
3. Functional equations over $\mathbb{R}$ where you use analytic properties of $f(x)$, such that continuity or differentiability or integrability.

2 Problems

2.1 Functional equations and the integers

Easier

1. (Putnam 1992) Show that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:
   (a) $f(f(n)) = n$ for all integers n
   (b) $f(f(n + 2) + 2) = n$ for all integers n
   (c) $f(0) = 1$.

Harder

2. Suppose $f : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ satisfies $f(n+1) > f(f(n))$ for all $n \geq 1$.
   (a) Show that $f(1)$ is the minimum value of f.
   (b) Show that $f(1) < f(2) < f(3) < \ldots$.
   (c) Show that $f(n) > n$ can never happen.
   (d) Deduce that $f(n) = n$ for all n.

2.2 Functional equations and algebraic manipulations

Easier

3. (Putnam 1971) Let $f(x)$ be a function defined on real numbers except 0 and 1. Find $f(x)$ knowing that it satisfies $f(x) + f(1 - 1/x) = 1 + x$.

Harder

4. (Putnam 1988) Show that there exists a unique function $f(x) : (0, \infty) \to (0, \infty)$ such that $f(f(x)) = 6x - f(x)$ for all $x > 0$.
5. (Putnam 1996) Let $c \geq 0$ be a constant. Give a complete description of the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbb{R}$.
6. (Putnam 2012) Let $f : [-1, 1] \to$ be a continuous function such that
   (i) $f(x) = \frac{2-x^2}{2} f\left(\frac{x^2}{2-x^2}\right)$ for every x in $[-1, 1]$,
   (ii) $f(0) = 1$, and
   (iii) $\lim_{x \to 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite.

   Prove that f is unique, and express $f(x)$ in closed form.

2.3 Functional equations and calculus

Easier

7. (Putnam 1971) Find all polynomials $P(x)$ such that $P(x^2 + 1) = P(x)^2 + 1$ and $P(0) = 0$.
8. (Putnam 1991) Suppose f and g are nonconstant differentiable real-valued functions on $\mathbb{R}$. Also suppose that for all x, y real,

$$f(x + y) = f(x)f(y) - g(x)g(y)$$
$$g(x + y) = f(x)g(y) + g(x)f(y)$$

   If $f'(0) = 0$ show that $f(x)^2 + g(x)^2 = 1$ for all x.
9. (Putnam 2010) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x + n) - f(x)}{n}$$

   for all real numbers x and all positive integers n.

Harder

10. (Putnam 2005) Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that
$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$
for all $x > 0$.

11. (Putnam 2000) Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for all x.

2.4 Extra problems

Easier

12. Suppose $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfies $f(f(n)) = n + 3$ for all integers $n \geq 0$.
    (a) Show that $f(n + 3) = f(n) + 3$.
    (b) Deduce that $f(3k) = 3k + f(0)$, $f(3k+1) = 3k + f(1)$ and $f(3k+2) = 3k + f(2)$ for all nonnegative integers k.
    (c) Show that $f(f(n)) \equiv n \pmod 3$ and conclude that either $f(x) \equiv x \pmod 3$ for at least one of $x \in \{0, 1, 2\}$.
    (d) Deduce that no such function $f(n)$ exists.

13. Suppose $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfies $f(xf(y)) = \frac{f(x)}{y}$ for all $x, y \in \mathbb{Q}_{>0}$.
    (a) Show that $f(f(y)) = f(1)/y$, that $f(f(1)) = 1$ and deduce that $f(1) = 1$.
    (b) Deduce that $f(f(y)) = 1/y$ and show that $f(1/y) = 1/f(y)$.
    (c) Show that $f(x/y) = f(x)/f(y)$.
    (d) Deduce that $f(xy) = f(x)f(y)$ for all x, y.
    (e) Can you find ONE example of such f?

14. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 1/2$ and there is some real α for which
$$f(x + y) = f(x)f(\alpha - y) + f(y)f(\alpha - x)$$
for all $x, y \in \mathbb{R}$.
    (a) Show that $f(\alpha) = 1/2$.
    (b) Show that $f(\alpha - x) = f(x)$ for all x.
    (c) Show that $f(x) = \pm 1/2$ for all x.
    (d) Show that in fact $f(x) = 1/2$ for all x.
    (e) Suppose we drop the assumption that $f(0) = 1/2$. Can you find a nonconstant solution to the functional equation?

15. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $xf(y) + yf(x) = (x + y)f(x)f(y)$. Show that for every $x \in \mathbb{R}$ we have $f(x) \in \{0, 1\}$. Can you show that f is an even function?

16. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x)f(y) = f(x - y)$ for all x, y and also suppose that f is not the 0 function. Show that $f(0) = 1$ and that for every $x \in \mathbb{R}$, $f(x) \in \{-1, 1\}$.

17. For each of the following functional equations find all continuous $f(x)$ that satisfy the equation:
    (a) $f(x + y) = f(x)f(y)$ with $f : \mathbb{R} \to (0, \infty)$.
    (b) $f(x + y) = f(x) + f(y) + f(x)f(y)$.
    (c) $f(xy) = f(x) + f(y)$ for $f : (0, \infty) \to \mathbb{R}$.
    (d) $f(xy) = xf(y) + yf(x)$ for $f : (0, \infty) \to \mathbb{R}$.

Harder

18. Determine all functions $f : [0, \infty) \to [0, \infty)$ satisfying the following properties: (a) $f(2) = 0$, (b) if $x \in [0, 2)$ then $f(x) \neq 0$, and (c) if $x, y \in [0, \infty)$ then $f(x + y) = f(xf(y))f(y)$.

19. Find the polynomials $P(X)$ such that $P(X + 1) = P(X) + 2X + 1$.

20. (Putnam 2016) Find all functions $f : (1, \infty) \to (1, \infty)$ with the following property: if $x, y \in (1, \infty)$ and $x^2 \leq y \leq x^3$ then $(f(x))^2 \leq f(y) \leq (f(x))^3$.

21. Determine the continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x)f(y)$. [Hint: Can you reduce to Exercise 17(a)?]

22. Find the continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$$

23. Determine the continuous functions $f : \mathbb{R} \to \mathbb{R}_{\neq 0}$ such that for all x, y,

$$f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)}$$