Math 43900 Problem Solving
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Lecture 13 Brainstorming

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In previous lectures we concentrated on different problem solving topics and techniques, as well as writing up clearly and concisely your solutions. Today we'll concentrate on brainstorming, by which I mean how to poke around effectively when stuck while keeping a clear eye.

1 Problems

1. (Putnam 1975) Suppose n is a sum of two triangular integers, i.e.,

$$n = \frac{a(a+1)}{2} + \frac{b(b+1)}{2}.$$

Show that $4n+1$ is a sum of two perfect squares and find x, y such that $4n+1 = x^2 + y^2$.

2. (Putnam 1975) For what real numbers b, c do both roots of $z^2 + bz + c = 0$ lie in the interior of the unit circle? Sketch the region in the bc-plane.

3. (Putnam 1983) Let p be an odd prime and $P(X) = 1 + 2X + 3X^2 + \cdots + (p-1)X^{p-2}$. Show that if $a \neq b \in \{0, 1, 2, \ldots, p-1\}$ then $p \nmid P(a) - P(b)$.

4. (Putnam 1976) Find all solutions to the equation $|p^r - q^s| = 1$ where p and q are primes and $r, s \geq 2$.

5. (Putnam 1977) Determine all real numbers x, y, z, w such that

$$x + y + z = w$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}.$$

6. (Putnam 1978) Let $0 < x_i < \pi$ for $i = 1, 2, \ldots, n$. Writing $x = \frac{1}{n}\sum x_i$ show that

$$\prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left(\frac{\sin x}{x}\right)^n.$$

7. (Putnam 1978) Consider n points in the plane. Show that fewer than $2n^{3/2}$ pairs of them are exactly unit distance apart.

8. (Putnam 1968) Suppose $f(x)$ is continuous and $\int_{-\infty}^{\infty} f(x)dx$ exists. Show that

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)dx.$$

9. (Putnam 2010) Given a positive integer n, what is the largest k such that the numbers $1, 2, \ldots, n$ can be put into k boxes such that the sum of the numbers in each box is the same? E.g., when $n = 8$ the example $(1, 2, 3, 6)$, $(4, 8)$, $(5, 7)$ shows that the largest k is at least 3.

10. (Putnam 2012) Let $\mathcal{S}$ be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

    (a) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in $\mathcal{S}$;

    (b) If $f(x), g(x)$ are in $\mathcal{S}$ then so are the function $f(x) + g(x)$ and $f(g(x))$;

    (c) If $f(x), g(x)$ are in $\mathcal{S}$ and $f(x) \geq g(x)$ for all $x \geq 0$ then the function $f(x) - g(x)$ is in $\mathcal{S}$.

    Prove that if $f(x), g(x)$ are in $\mathcal{S}$ then so is the function $f(x)g(x)$.

11. (Putnam 2014) Let A be the $n \times n$ matrix whose entry on row i and column j is $1/\min(i, j)$. Compute $\det A$.

12. (Putnam 1960) Consider the sequence $(a_n)_{n \geq 0}$ defined by $a_0 = 0$ and $a_{n+1} = 1 + \sin(a_n - 1)$ for $n \geq 0$. Compute
$$\lim_{n \to \infty} \frac{a_0 + a_1 + \cdots + a_n}{n}.$$

13. (Putnam 1961) The set of pairs of positive reals (x, y) such that $x^y = y^x$ form the straight line $y = x$ and a curve. Find the point at which the curve cuts the line.

14. (Putnam 1963) Find all twice differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x)^2 - f(y)^2 = f(x + y)f(x - y),$$
for all reals x, y.

15. (Putnam 1967) Find the smallest positive integer n such that we can find a polynomial $nx^2 + ax + b$ with integer coefficients and two distinct roots in the interval $(0, 1)$.