

Math 43900 Problem Solving
Fall 2023
Lecture 14 Which problems do I choose?

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1. (Putnam 1985) Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

- (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and
(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

2. (Putnam 1985) Let $I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) dx$. For which integers m , $1 \leq m \leq 10$ is $I_m \neq 0$?
3. (Putnam 1986) Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?
4. (Putnam 1987) The sequence of digits

123456789101112131415161718192021...

is obtained by writing the positive integers in order. If the 10^n -th digit in this sequence occurs in the part of the sequence in which the m -digit numbers are placed, define $f(n)$ to be m . For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

5. (Putnam 1987) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)} dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

6. (Putnam 1988) Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

7. (Putnam 1990) Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

8. (Putnam 1992) Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

- (i) $f(f(n)) = n$, for all integers n ;
- (ii) $f(f(n + 2) + 2) = n$ for all integers n ;
- (iii) $f(0) = 1$.

9. (Putnam 1998) Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \dots 11.$$

Find the thousandth digit after the decimal point of \sqrt{N} .

10. (Putnam 2001) You have coins C_1, C_2, \dots, C_n . For each k , C_k is biased so that, when tossed, it has probability $1/(2k + 1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n .
11. (Putnam 2001) For each integer m , consider the polynomial

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

12. (Putnam 2008) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?