Math 43900 Fall 2023 Problem Solving
Lecture 2: Induction

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These problems are taken from the textbook, from past Putnam competitions, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

Mathematical induction

Induction where you know what you need to show

1. Show that $3^n \geq n^3$ for all positive integers n. (AG 14)

2. For an integer n define $f(n)$ by the following rules: $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(n)+1$. Show that $f(n)$ is the number of 1s in the binary representation of n.

3. Prove for all positive numbers the identity

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

4. Let $n \geq 6$ be an integer. Show that
$$\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n.$$
(AG 15)

5. Show that for any $n \geq 4$ an isosceles triangle with one angle of 120° can be dissected into n triangles similar to it. (AG 26)

6. Show that every positive integer can be written in the form $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$ for some $n \geq 1$ and some choice of signs.

7. Prove that for any positive integer $n \geq 2$ there exists a positive integer m that can be written simultaneously as a sum of $2, 3, \ldots, n$ squares of nonzero integers.

8. Let a_n be the number of ways to tile a $1 \times n$ strip by 1×1 and 1×3 tiles.
   (a) Show that $a_n < 1.5^n$.
   (b) Which happens more often, that a_n is odd or even?

9. Consider the Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$. Show the following identities by induction
   (a) $F_n < 1.75^{n-1}$ for $n \geq 2$,
   (b) $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$,
   (c) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.
   (d) $F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}$.

10. Show that if $a_1, \dots, a_n > 0$ then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Induction where you need to figure out what you want to prove

If you don't know what precise statement to prove by induction, you should try some small cases to guess the statement you'd like to prove.

11. Find a formula for the sum of the first n odd numbers.

12. You aim to spell Notre Dame in the following diagram, by starting at the top left and only going down and/or right. How many ways can you do this?

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NOTREDAME
OTREDAME
TREDAME
REDAME
EDAME
DAME
AME
ME
E
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13. A number of lines, any three of which don't pass through the same point, partition the plane into disjoint region. Show that you can color these regions with two colors, such that no two regions which share an edge have the same color.

14. Show that the interior angles of a convex n-gon always add up to the same value.

15. Find a formula for x_n knowing that $x_1 = \frac{5}{2}$ and $x_{n+1} = x_n^2 - 2$ for all $n \geq 1$.

16. Define the polynomials $P_n(X)$ for $n \geq 0$ by $P_0(X) = 1$, $P_n(0) = 0$ for $n \geq 1$ and

$$P'_{n+1}(X) = (n+1)P_n(X+1).$$

Factor $P_{100}(1)$. (Putnam 1985)

17. Find a closed formula for $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}^n$. (Useful for differential equations!)

Due next week

Write

Please write out clearly and concisely one of the following:

1. one problem from the ones I explained in class and one problem of your choosing that I did not cover in class OR

2. two problems that I did not cover in class.