Math 43900 Fall 2022 Problem Solving
Lecture 3: Calculus

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These problems are taken from the textbook, from Engel's *Problem solving strategies*, from Ravi Vakil's Putnam seminar notes and from Po-Shen Loh's Putnam seminar notes.

1 Overview

Today we'll concentrate on derivatives and integrals. Calculus is a vast topic but is also the one where you have seen most examples in your previous courses. You will rarely need any new calculus technique that you haven't seen before. The difficulty, on the contrary, is to patch together all the things you know to obtain a solution.

Again, while cleverness will take you a long way in problem solving calculus, there's no place for being squeamish about algebraic manipulations. Finally, a comment about rigor: when **solving** problems, don't worry about being rigorous at first. Better to have a complete solution that's missing steps or perhaps is not as rigorous as it should, than to have a completely rigorous write-up of nothing much.

Calculus problems that you see in competitions, much like in the real world, tend to combine ideas from many topics. You could have a derivative problem for maximization that involves limits of integrals. I identified 3 rough types, although the textbook has many more collections of calculus related problems in §3:

1. Functions and their analytic properties, e.g., continuity, etc.
2. Computing integrals or perhaps limits of integrals.
3. Calculus problems in geometry.

2 Problems

2.1 Continuity and derivatives

Easier

1. (Putnam 1986) Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.
2. Show that every continuous function $f : [a, b] \to [a, b]$ has a fixed point, i.e., $f(c) = c$ for some $c \in [a, b]$. [Hint: the intermediate value theorem.]
3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all x, y. Show that (a) f is injective and (b) f is surjective.

4. (Putnam 2022) Determine all ordered pairs of real numbers (a, b) such that the line $y = ax + b$ intersects the curve $y = \ln(1 + x^2)$ in exactly one point.

Harder

5. (Putnam 2021) For every positive real number x, let

$$g(x) = \lim_{r \to 0} \left((x+1)^{r+1} - x^r\right)^{1/r}.$$

Find $\lim_{x\to\infty} g(x)/x$.

6. Let $P(X) = a_1X + a_2X^2 + \cdots + a_nX^n$ and $Q(X) = \sum_{k=1}^{n} a_kX^k/(2^k - 1)$ with $a_1, \ldots \in \mathbb{R}$. Show that if $Q(2^{n+1}) = Q(1) = 0$ then $P(X)$ has a positive root $< 2^n$. [Hint: $2^k + 2^{2k} + \cdots + 2^{kn} = \dfrac{2^{k(n+1)} - 1}{2^k - 1}$.]

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $g(x) = f(x) \int_0^x f(t)\,dt$. Show that if $g(x)$ is non-increasing then f is the 0 function. [Hint: Write $g(x)$ as the derivative of a conveniently simple function.]

8. (Putnam 2019) f is twice differentiable and satisfies

$$xf_x + yf_y = xy\ln(xy)$$
$$x^2f_{xx} + y^2f_{yy} = xy.$$

Compute

$$\min_{s\geq 1} f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s).$$

2.2 Integrals

Easier

9. Compute $\int (x^6 + x^3)\sqrt[3]{x^3 + 2}\,dx$. [Hint: put a factor of x inside the cube root.]

10. (Putnam 1979) Given $0 < \alpha < \beta$, find $\lim_{\lambda\to 0} \left(\int_0^1 (\beta x + \alpha(1-x))^\lambda\,dx\right)^{1/\lambda}$.

Harder

11. (Putnam 1980) Compute $\int_0^{\pi/2} \dfrac{dx}{1 + \tan^{\sqrt{2}}(x)}$.

12. Compute $\int \dfrac{x^n}{1 + x + x^2/2! + \cdots + x^n/n!}dx$. [Hint: What is the derivative of the denominator?]

13. (Putnam 2019) Is there a nonzero continuous function $f(x, y, z)$ such that the integral over (the surface of) any sphere of radius 1 is 0?

2.3 Geometry

Easier

14. Curves A, B, C and D are defined in the plane as follows:
$$A = \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \qquad B = \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\},$$
$$C = \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \qquad D = \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}.$$
Prove that $A \cap B = C \cap D$.

15. (Putnam 1984) V is the pyramidal region $x, y, z \geq 0, x+y+z \leq 1$. Evaluate
$$\int_V xy^9z^8(1-x-y-z)^4\,dx\,dy\,dz.$$

Harder

16. (Putnam 1983) Let T be the triangle with vertices $(0,0)$, $(a,0)$, and $(0,a)$. Find
$$\lim_{a\to\infty} a^4e^{-a^3}\int_T e^{x^3+y^3}\,dx\,dy.$$
[Hint: Rotate the triangle by 45 degrees, then use L'Hôpital.]

17. (Putnam 2015) Let A and B be points on the hyperbola $xy = 1$. Let P be a point on the arc AB of the hyperbola such that the triangle APB has largest area. Show that the area bounded by the hyperbola and the chord AP is the same as the area bounded by the hyperbola and the chord BP.

2.4 Extra problems

Easier

18. Compute $\displaystyle\int \frac{x+\sin x-\cos x-1}{x+e^x+\sin x}\,dx$. [Hint: add and subtract e^x on top.]

19. Solve $2^x = x^2$ for $x > 0$.

20. Determine the largest value of $|z^3 - z + 2|$ as z varies among the complex numbers such that $|z| = 1$. [Hint: use brute force, i.e., write $z = x + iy$ with $x^2 + y^2 = 1$ and reduce to a simple calc-1 maximization in terms of x. This really is only computations.]

21. (Putnam 2002) Let k be a fixed positive integer. The n-th derivative of $\dfrac{1}{x^k-1}$ has the form $\dfrac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

22. Let $f : [0,\infty) \to \mathbb{R}$ be an increasing continuous function such that $f(0) = 0$. You rotate the graph of $f(x)$ over the interval $[0,a]$ around the y-axis to get the solid R_a, which looks like a dish. Assume that for each a, the volume of R_a is also equal to the volume of water the dish R_a can hold. Find $f(x)$. [Hint: Writing the equality of volumes yields a differential equation that you can solve.]

23. Let $f : [0, \infty) \to \mathbb{R}$ be an increasing continuous function such that $f(0) = 0$. You rotate the graph of $f(x)$ over the interval $[0, a]$ around the y-axis to get the solid R_a, which looks like a dish. Assume that for each a, the volume of R_a is also equal to the volume of water the dish R_a can hold. Find $f(x)$. [Hint: Writing the equality of volumes yields a differential equation that you can solve.]

24. Show that every convex polygon can be divided by two perpendicular lines into four regions of equal area.

Harder

25. (From last year's VTRMC and also in the textbook exercise 459 and also on the Putnam in 2005) Compute $\displaystyle\int_1^2 \frac{\ln(x)}{2 - 2x + x^2} dx$.

26. Find all continuous functions $f : \mathbb{R} \to [1, \infty)$ for which there exists $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 1}$ such that
$$f(x)f(2x) \cdots f(nx) \leq an^k$$
for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 1}$. [Hint: Take log and use Riemann sums to estimate $\int_0^1 \ln f(x) dx$.]

27. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\displaystyle\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0, 1)$,
$$\left| \int_0^\alpha f(x)dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

28. Compute $\displaystyle\lim_{n\to\infty} \left(\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right)$. [Hint: Use Riemann sums.]

29. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that for every $x \in [0, 1]$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{f(x^n)}{2^n}$ converges. [Hint: At first don't try to be rigorous.]

Due next week

Write

Please write out clearly and concisely two problems.

Read

In preparation for next class, please look over section on polynomials (§2.2) in the textbook.

Attempt

Please look over the problems from the following lecture. This way you can ask me questions and we can discuss solutions in class.