Math 43900 Fall 2022 Problem Solving
Lecture 4: Polynomials

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These problems are taken from the textbook, from Ravi Vakil's Putnam seminar notes, from David Galvin's problems and from Po-Shen Loh's Putnam seminar notes.

Polynomials

Useful facts

1. If $P(X)$ has root α then $X - \alpha \mid P(X)$, i.e., $P(X) = (X - \alpha)Q(X)$ for a polynomial $Q(X)$. The root α is a double root, i.e., it appears twice in the list of roots, if and only if $P(\alpha) = P'(\alpha) = 0$.

2. If a polynomial with coefficients in $\mathbb{C}$ has degree at most n and at least $n + 1$ roots, it must be the 0 polynomial. In particular, a polynomial with infinitely many roots must be the 0 polynomial. A variant is that if P, Q are complex polynomials with $P(z) = Q(z)$ for infinitely many values of z then $P = Q$.

3. If $P(X)$ and $Q(X)$ have the same (complex) roots then they differ by a scalar. In particular, if they have the same leading coefficient then $P = Q$.

4. Remember from the quadratic formula that if $X^2 + aX + b = 0$ has roots α and β then $\alpha + \beta = -a$ and $\alpha\beta = b$. If $P(X) = X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_{n-1}X + a_n$ has roots $\alpha_1, \ldots, \alpha_n$ then for $1 \leq r \leq n$

$$(-1)^r a_r = \sum_{i_1 < i_2 < \ldots < i_r} \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_r} (= s_r)$$

   which specializes to $-a_1 = \sum_i \alpha_i (= s_1)$, $a_2 = \sum_{i<j} \alpha_i\alpha_j (= s_2)$, $-a_3 = \sum_{i<j<k} \alpha_i\alpha_j\alpha_k (= s_3)$ and so on until $(-1)^n a_n = \prod \alpha_i (= s_n)$. The s_k are called the **elementary symmetric polynomials** in the roots.

5. If A and B are two polynomials then you can divide with remainder: $A(X) = B(X) \cdot Q(X) + R(X)$ with either $R(X) = 0$ or $\deg R < \deg B$. Using divisibilities you can show that the gcd of A and B is the same as the gcd of B and R and then compute the gcd sequentially. We write (A, B) for the gcd.

6. Gauss's lemma: If A and B are integer polynomials and A/B is a polynomial (necessarily with rational coefficients) then it is an integer polynomial. In other words if $B \mid A$ as rational polynomials then $B \mid A$ as integral polynomials.

7. If a matrix has entries which are polynomials then the determinant of the matrix is also a polynomial. You can show this by induction using the fact that a determinant can be expanded in terms of rows and minors.

8. The important Eisenstein irreducibility criterion, which we'll prove when we do modular arithmetic. Suppose $P(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1}X + a_n$ is an integral polynomial and p is a prime number such that $p \mid a_1, a_2, \ldots, a_n$ but $p^2 \nmid a_n$. Then $P(X)$ is an irreducible polynomial.

9. Finally an input from Galois theory that's useful: If a rational (or real or complex) polynomial $P(x_1, x_2, \ldots, x_n)$ doesn't depend on the ordering of the variables $x_1, \ldots, x_n$, i.e., no matter how you permute them the final expression is the same, then $P(x_1, \ldots, x_n)$ can be written as a polynomial rational (or real or complex) polynomial $Q(s_1, \ldots, s_n)$ where s_k are the elementary symmetric polynomials. E.g., $x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 = s_1s_2 - 3s_3$ (check this!).

Problems with roots

Easier

1. (Putnam 2005) Find a non-zero polynomial $P(X,Y)$ such that $P(\lfloor t \rfloor, \lfloor 2t \rfloor) = 0$ for all real numbers t. (Here $\lfloor t \rfloor$ indicates the greatest integer less than or equal to t.)

2. (Putnam 1985) Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial
$$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$
has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

3. (Putnam 1992) Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let
$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$
for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

4. (Putnam 1979) Let F be a finite field with an odd number n of elements. Suppose $x^2 + bx + c$ is an irreducible polynomial over F. For how many elements $d \in F$ is $x^2 + bx + c + d$ irreducible?

Harder

5. (Putnam 1991) Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

   (a) $p(r_i) = 0, \qquad i = 1, 2, \ldots, n$, and
   (b) $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0 \qquad i = 1, 2, \ldots, n-1$,

   where $p'(x)$ denotes the derivative of $p(x)$.

6. If $P(X)$ is a real polynomial whose roots are all real and distinct and different from 0 show that $XP'(X) + P(X)$ is a real polynomial with distinct real roots which are different from 0. As a follow-up: show that $XP''(X) + 3XP'(X) + P(X)$ has distinct real roots. [Hint for the follow-up: apply the first part twice.]

Problems with divisibilities

Easier

7. Show that in the product $(1-X+X^2-X^3+\cdots+X^{100})(1+X+X^2+X^3+\cdots+X^{100})$ when you expand and collect terms X only appears to even exponents.

8. (Putnam 2016) Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \left.\frac{d^j}{dx^j}p(x)\right|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

9. Find all polynomials $P(X)$ satisfying $(X+1)P(X) = (X-2)P(X+1)$.

Harder

10. Let $a_1 < a_2 < \ldots < a_n$ be integers. Show that $(X-a_1)(X-a_2)\cdots(X-a_n)-1$ is irreducible in $\mathbb{Z}[X]$. [Hint: If it factors as $P(X)Q(X)$ what are the roots of $P+Q$?]

11. Suppose p is a prime $\equiv 3 \pmod 4$. Show that $(X^2+1)^n + p$ is irreducible over $\mathbb{Z}$. [Hint: the condition on p implies that X^2+1 has no roots mod p.]

12. Let $P(X) \in \mathbb{Z}[X]$ be an irreducible polynomial such that $|P(0)|$ is not a perfect square. Show that $P(X^2)$ is also irreducible.

Extra problems

Easier

13. Show that the polynomial $X^n - 2$ is irreducible in $\mathbb{Z}[X]$.

14. Suppose p is a prime. Show that $P(X) = X^{p-1} + X^{p-2} + \cdots + X + 1 = \dfrac{X^p-1}{X-1}$ is an irreducible polynomial. [Hint: Look at $P(X+1)$ and apply the Eisenstein irreducibility criterion.]

15. Suppose $P(X)$ is a monic polynomial with integer coefficients. Show that if $P(X)$ has a rational root α then α is in fact integral. [Roots of such polynomials are called algebraic integers.]

16. For which real values of p and q are the roots of the polynomial $X^3 - pX^2 + 11X - q$ three consecutive integers?

Harder

17. (Useful) Show that if $m \mid n$ then $X^m - 1 \mid X^n - 1$. Also show that if $m \mid n$ are odd then $X^m + 1 \mid X^n + 1$. As a follow-up: show that if m and n are positive integers with gcd d then the polynomials $X^m - 1$ and $X^n - 1$ have gcd $X^d - 1$. [Hint: Show that if $m = nq + r$ is division with remainder then $X^m - 1 = (X^n - 1)Q(X) + X^r - 1$ is division with remainder.]

18. (Putnam 2017) Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

19. (Putnam 1986) Let $a_1, a_2, \dots, a_n$ be real numbers, and let $b_1, b_2, \dots, b_n$ be distinct positive integers. Suppose that there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \dots, b_n$ and n (but independent of $a_1, a_2, \dots, a_n$).

20. Find all complex numbers a, b such that $|z^2 + az + b| = 1$ for all complex numbers z with $|z| = 1$.

21. Let $P(X) = X^n + a_1X^{n-1} + \cdots + a_{n-1}X + a_n$. If $a_1 + a_3 + a_5 + \cdots$ and $a_2 + a_4 + \cdots$ are real numbers show that $P(1)$ and $P(-1)$ are real numbers as well. As a follow-up: let $\alpha_1, \dots, \alpha_n$ be the roots of $P(X)$ and suppose that $Q(X) = X^n + b_1X^{n-1} + \cdots b_{n-1}X + b_n$ has roots $\alpha_1^2, \dots, \alpha_n^2$. Show that $b_1 + b_2 + \cdots + b_n$ is a real number.

22. For which values of $n \geq 1$ do there exist polynomials $P(X)$ of degree n satisfying:
    (a) $P(k) = k$ for $1 \leq k \leq n$,
    (b) $P(0)$ is an integer, and
    (c) $P(-1) = 2017$?

Due next week

Write

Please write out clearly and concisely two problems.

Read

In preparation for next class, please look over section on the pigeonhole principle (§1.3) in the textbook.

Attempt

Please look over the problems from the following lecture. This way you can ask me questions and we can discuss solutions in class.