Math 43900 Fall 2023 Problem Solving
Lecture 5: The pigeonhole principle

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Combinatorial

Easier

1. Show that in the sequence of powers of 2023, the last 3 digits form a periodic sequence.
2. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there exist two distinct integers in A which add up to 104.
3. Show that at any party there are two people who know exactly the same number of people at the party.
4. Let α be an irrational number. Show that the set of fractional parts $\{n\alpha\} = n\alpha - \lfloor n\alpha \rfloor$ is dense in $[0, 1)$ (that is, any open interval within $[0, 1)$, no matter how small, contains one of these fractional parts).

Harder

5. Prove that for any sequence of digits $a_1 \ldots a_k$ there exists a power of 2 that begins with $a_1 \ldots a_k$ in decimal expansion.
6. (Putnam 1980)
   (a) Prove that there exist integers a, b, c not all zero and each of absolute value less than one million such that
   $$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$
   (b) Let a, b, c be integers, not all 0, and each of absolute value less than one million. Prove that
   $$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.$$
7. (Putnam 1993) Let $x_1, \ldots, x_{19}$ be positive integers ≤ 93. Let $y_1, \ldots, y_{93}$ be positive integers ≤ 19. Prove that there exists a nonempty sum of the x_i's equal to a sum of some y_j's.

Geometric

Geometrically the pigeonhole principle states that if you have a number of subsets of a bigger geometric set with total length/area/volume larger than the length/area/volume of the bigger set then at least two of the smaller subsets must intersect.

Easier

8. (Putnam 2002). Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.
9. (Putnam 1971) A set S of nine points with integral coordinates is given in 3D space. Show that there exists a point with integral coordinates in the interior of one of the line segments joining two points in S.

Harder

10. Inside a circle of radius 4 are 45 points. Show that you can find two of these points at most $\sqrt{2}$ apart. [Hint: Draw circles around each point.]

11. Fifty-one points are placed in a square of side length 1. Prove that there is a circle of radius 1/7 that contains three of the points.

12. (Putnam 1990) Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$. [Hint: A lattice point is a point with integral coordinates. A polygon whose vertices are lattice points has area $I + B/2 - 1$ where I is the number of interior lattice points and B is the number of lattice points on the boundary. This is Pick's theorem.]

Extra problems

Easier

13. Show that in any group of 6 people you can find 3 who know each other or 3 who are strangers to each other. (This is a classical example in Ramsey theory, $R(3,3) = 6$. Erdős claimed that if aliens attack us and threaten to kill us unless we tell them $R(5,5)$ we should invest all of Earth's resources to compute $R(5,5)$, but if they demand $R(6,6)$ instead we should invest all of our resources to destroy them.)

14. Given n integers, prove that some nonempty subset of them has sum divisible by n.

Harder

15. (Erdős) Let $A \subset \{1, 2, \ldots, 2n\}$ be a set of $n+1$ integers. Prove that some element of A divides another.

16. (Putnam 2000). Let a_j, b_j, c_j be integers for $1 \leq j \leq n$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $\frac{4}{7}n$ values of j between 1 and n.

17. (Putnam 1994) Let A and B be 2 by 2 matrices with integer entries such that $A, A+B, A+2B, A+3B$ and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

18. (IMO 1972) Prove that from a set of ten distinct two-digit numbers, it is possible to select two non-empty disjoint subsets whose members have the same sum.

Due next week

Write

Please write out clearly and concisely two problems.

Read

In preparation for next class, please look over section on invariants (§1.5) in the textbook.

Attempt

Please look over the problems from the following lecture. This way you can ask me questions and we can discuss solutions in class.