Introduction to Financial Mathematics

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Spring 2023

Financial Instruments

1. Cash: deposit, loans
2. Fixed income assets: bonds, pensions, mortgages, annuities
3. Variable assets: stocks, real estate, foreign currency
4. Contract assets: futures, derivatives

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- How good is an investment?
- How to protect an investment?
- How to measure risk?

What is this course?

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- An introduction to the **math** behind pricing and understanding basic financial assets.
- A fundamental exploration of loans and interest rate changes.
- A first step towards portfolio construction and hedging.

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What this course is not

This course is not a test-prep course.

In particular, many times during class we will use spreadsheets even if they are not allowed on the actuary exam.

Course Description

1. Textbook: Mathematics of Investment and Credit, Samuel A. Broverman, 6th or 7th ed.
2. Homework: There will be weekly problem sets.
3. Final grade: 40% homework, 15% each of two midterms, 30% final exam.
4. Office hours: Hurley 275, TBD

What is the value of an investment?

Definition

$A(t)$ is the value of an asset or investment at time t, measured in years, months, days, seconds, etc.

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1. Understand $A(t)$.
2. Predict $A(t)$.
3. Increase $A(t)$.
4. Minimize risk.

Simpler goal: measure changes to $A(t)$.

Definition

The **yield rate** of an investment between time t_1 and t_2 is

$$\frac{A(t_2) - A(t_1)}{A(t_1)} = \frac{A(t_2)}{A(t_1)} - 1.$$

Market example: Bitcoin

Example

What is $A(t)$ for Bitcoin over the past year? What is the yield rate?

Market example: Bitcoin

Example

What is $A(t)$ for Bitcoin over the past half a year? What is the yield rate?

Lecture 2 January 20, 2023

United States Economic Data

Data	2022	2023
GDP	\$20.9 trillion	\$25 trillion
Budget	\$6.8 trillion	\$6.272 trillion
Borrowing rate	0.05%, 1.5%	4.33%, 3.83%
Bank lending rate 3.25%	7.27%	
Mortgage rate	4%	6.89%
Inflation rate	6.8%	6.5%
Stock market (S&P500)	24%	-19%

Basic example: interest paying account

Example

Invest C into an account that pays interest i per period. What is $A(t)$, after t periods?

Accumulated factor

Definition

The **accumulated factor** of an investment is

$$a(t) = \frac{A(t)}{A(0)}.$$

Example

Interest paying account.

Accumulated factor

Definition

The **accumulated factor** of an investment is

$$a(t) = \frac{A(t)}{A(0)} \qquad a(t_1, t_2) = \frac{A(t_2)}{A(t_1)}$$

Example

Interest $i = 1\%$ per year, after 50 years

Accumulated factor

Definition

The **accumulated factor** of an investment is

$$a(t) = \frac{A(t)}{A(0)} \qquad a(t_1, t_2) = \frac{A(t_2)}{A(t_1)}$$

Example

What interest rate should I seek if I wanted to double the investment in 10 years? (7.1%)

Algebra

$$e^x \approx 1 + x$$

$$\ln(1 + x) \approx x$$

Keeping track of accounting vs time value of money

Time value of money

Given an accumulation factor $a(t)$ (e.g., fixed bank account, investment, market conditions, etc) the value of X at time t_1 when considered at time t_2 is $X \cdot a(t_1, t_2)$.

Example

You invest \$5000 on January 1st at a monthly interest rate of 2%. You withdraw \$1000 on June 1st and September 1st. How much money do you have on December 31st?

Keeping track of accounting vs time value of money

Time value of money

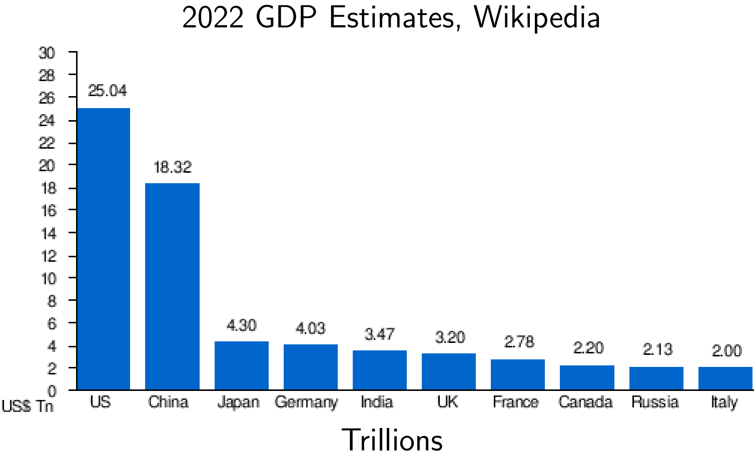
Given an accumulation factor $a(t)$ (e.g., fixed bank account, investment, market conditions, etc) the value of X at time t_1 when considered at time t_2 is $X \cdot a(t_1, t_2)$.

The mathematical fiction of time value of money is useful precisely because it allows one to shift the time frame as necessary:

Example

You invest X on January 1st at a monthly interest rate of 2%. You withdraw \$1000 on June 1st and September 1st. On December 31st you have \$5000 in the account? What is X?

Lecture 3 January 23, 2023



Recall from last lecture

Time value of money

Compound Interest

Example

Suppose I borrow \$1000 from a loan shark with 10% per week, and I repay after 3 days. What do I owe, assuming the loan shark is financially literate and fair? (\$1041.7)

Compound Interest

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Definition

If i is the interest rate per period, **compound interest** refers to a method of computing the accumulated factor as

$$a(t) = (1+i)^t$$

even if t is not an integer.

Accumulated factor as a way between interest rates

Definition

The **effective annual interest rate** of an investment is

1. the yield rate of the investment over a year OR, equivalently,
2. the **compound** annual interest rate i which yields the same accumulated factor as the investment over the period of the investment.

Example

What is the effective annual interest rate of 2% monthly? (26.8%)

Effective annual IR as a way to compare IRs

Example

Which one is a better investment: $\frac{3}{4}\%$ over 17 days or 3% over 67 days?

An example: Facebook

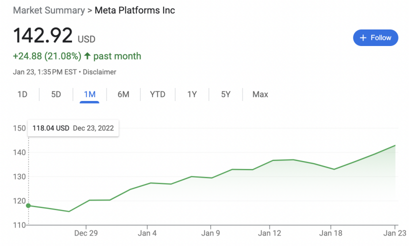


Figure: A

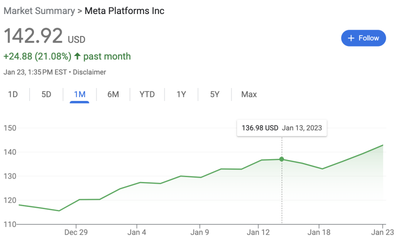


Figure: B

Example

A bought Meta stock one month ago, B 10 days ago. Both sold today. Who made a better investment? ($A = 892\%$ or $A = 850\%$, $B = 370\%$)

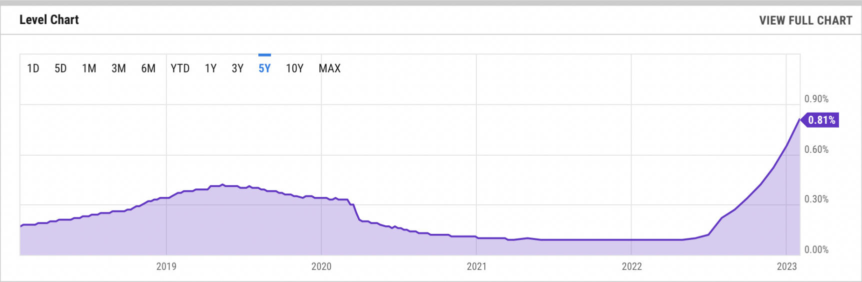
Lecture 4 January 25, 2023

Certificate of Deposit Interest Rates

US 6-Month CD Rate

0.81% for Jan 2023

Overview   Interactive Chart



Recall from last lecture

Effective annual interest rate and compound interest.

Average annual interest/yield rate

Definition

The **average annual interest rate** (or yield rate, or rate of return) is the compound annual interest rate which yields the same accumulated factor as a (multi-year) investment.

Example

An investor earns 10% profit one year, and loses 2% each of the following 3 years. How good was the investment? (0.87%)

Average annual interest/yield rate

Definition

The **average annual interest rate** (or yield rate, or rate of return) is the compound annual interest rate which yields the same accumulated factor as a (multi-year) investment.

Say $i_1, i_2, \ldots, i_n$ are annual interest/yield rates over a period of n years. The average annual IR is i means that

$$(1 + i_1)(1 + i_2) \cdots (1 + i_n) = (1 + i)^n$$

Average annual IR example

Example

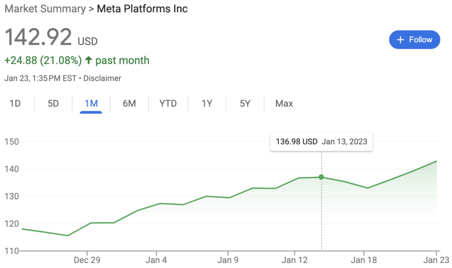
The S&P 500 index had the following returns over the last 5 years. What is the average annual interest rate/yield rate/rate of return? (17.7%) The average annual return over the last 6 years was 10.5%. What was the return in 2022? (-19.44%)

Year	Yield rate
2017	21.8%
2018	-4.4%
2019	31.5%
2020	16.3%
2021	26.9%
2022	$?$

Computing interest for part of periods

Compound Interest	**Simple Interest**
$a(t) = (1+i)^t$	$a(t) = 1 + it$

Back to Facebook

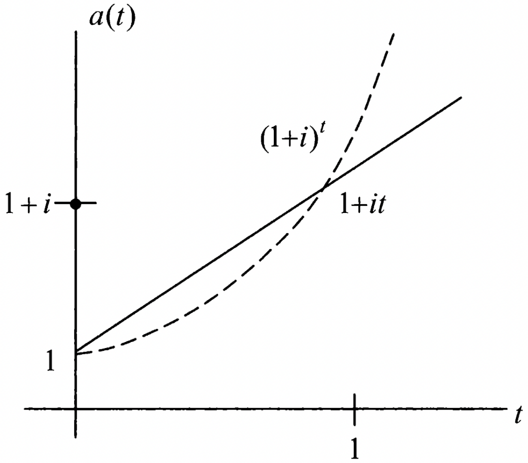


Example

What is the effective annual interest rate for Meta's performance over the last month, computed using **simple interest**? (253% or 248%)

Compound vs Simple Interest

Why this dramatic difference?



Lecture 5 January 27, 2023

How much money is there? (In trillions)

	USD	EUR
Cash (M0)	2.1	1.3
"Narrow money" (M1)	40	10
"Broad money" (M2/M3)	90	15
All financial assets	1300	

Recall from last lecture

Compound vs simple interest

A more involved example

1.1.9.a

Jones invests 100,000 in a 180-day short term guaranteed investment certificate at a bank, based on simple interest at annual rate 7.5%. After 120 days, interest rates have risen to 9% and Jones would like to redeem the certificate early and reinvest in a 60-day certificate at the higher rate. In order for there to be no advantage in redeeming early and reinvesting at the higher rate, what early redemption penalty (from the accumulated book value of the investment certificate to time 120 days) should the bank charge at the time of early redemption?

(278.93)

Time value of money: Present Value

We previously saw that to find the value of money in the future we multiply by the accumulated factor:

Time value of money

The value of X now at some time t in the future is $X \cdot a(t)$.

Definition

The **present value** of C at time t is

$$\boxed{\mathrm{PV}(C) = \mathrm{PV}(C \text{ at time } t) = \frac{C}{a(t)}}$$

An example

Example

What should be worth more to me, given that the interest rate (assumed constant) is 7%: \$120 in 2 years, or \$140 in 4 years, or \$160 in 7 years?

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What should be worth more to me, given that the interest rate (assumed constant) is 7%: \$120 in 2 years, or \$140 in 4 years, or \$160 in 7 years?

$$a(2) = (1+i)^2 = 1.07^2 = 1.145$$
$$a(4) = (1+i)^4 = 1.311$$
$$a(7) = (1+i)^7 = 1.606$$

$$\mathrm{PV}(120 \text{ at time } 2) = \frac{120}{a(2)} = 104.80$$
$$\mathrm{PV}(140 \text{ at time } 4) = \frac{140}{a(4)} = 106.79$$
$$\mathrm{PV}(160 \text{ at time } 7) = \frac{160}{a(7)} = 99.63$$

Present Value vs Time Value

Remark

The "present" in "present value" can be anything, it doesn't have to be the arbitrarily chosen value $t = 0$.

Previous example: value at 120 days

1.1.9.a

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(278.93)

Present value for compound interest

Accumulated factor & present value

$$a(t) = (1+i)^t \quad \mathrm{PV}(C \text{ at time } t) = \frac{C}{(1+i)^t}$$

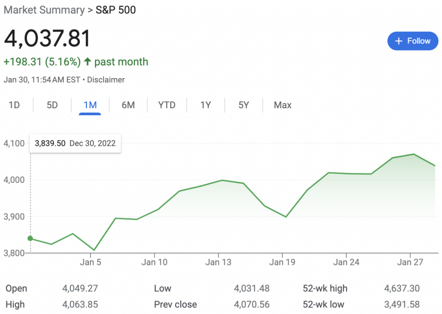
A little algebra magic:

$$\mathrm{PV}(C \text{ at time } t) = \frac{C}{(1+i)^t} = C \cdot (1+i)^{-t} = C \cdot a(-t)$$

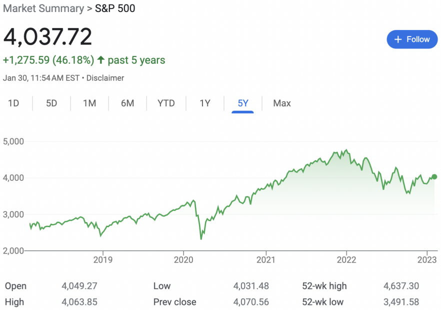
Time value with compound interest

To find the value of money t periods in the future multiply by $a(t) = (1+i)^t$. To find the value of money t periods in the past multiply by $a(-t) = (1+i)^{-t}$.

Lecture 6 January 30, 2022



Stock Market S&P 500



Recall from last lecture

Present Value

Terminology for PV

Suppose we are operating with **compound interest**.

Definition

The **present value factor** is $\nu = \dfrac{1}{1+i}$.

$$\mathrm{PV}(C \text{ at time } t) = \frac{C}{(1+i)^t} = C \cdot \left(\frac{1}{1+i}\right)^t = C \cdot \nu^t.$$

Lottery winnings

If you win the lottery you are typically presented with two ways of cashing in: either one lump sum, or periodic smaller sums.

Example

You won the lottery. You are presented with two options:

1. Get \$4 million now, or
2. Get \$1 million now and again \$1 million each year thereafter, for 4 years.

If $i = 3\%$, which choice is financially more advantageous? (Second) For what interest rate are the two choices financially the same? (12.6%)

Equations of value

Definition

Given a series of cashflows, an **equation of value** at time t is a computation of the **value** at time t of the whole series of cashflows. This is made up of

1. The value at time t of all cashflows occurring prior to t and
2. The present value at time t of all future cashflows.

Rebates

Example

You purchase a TV for $900 with a mail-in rebate of $150, payable in 8 weeks.

1. What percentage discount does the rebate's value represent? (16.67%)
2. What is the effective percentage discount assuming an annual interest rate of 3%? (PV 149.32, 16.59%)

Tax withholdings

Every employed person has to complete a W-4 form specifying what percentage of the salary is withheld for tax purposes. This percentage depends on marital status, number of dependents, etc.

Example

A is hired at the end of July and files a W-4 form which results in monthly withholdings of $2000 for federal income taxes. A knows they accidentally chose a higher withholding bracket and, in fact, their monthly withholdings should be around $1500. A has two options: either go back to HR and fix the mistake now, at the end of July, or wait until the end of December, and be reimbursed for the overpay at the end of April. If $i = 3\%$, what is A's financial incentive to fix the mistake now? ($\$36.42$)

Lecture 7 February 1, 2023

MONTHLY STATEMENT OF THE PUBLIC DEBT OF THE UNITED STATES

DECEMBER 31, 2022

(Details may not add to totals)

TABLE I -- SUMMARY OF TREASURY SECURITIES OUTSTANDING, DECEMBER 31, 2022

(Millions of dollars)

	Amount Outstanding		
	Debt Held By the Public	Intragovernmental Holdings	Totals
Marketable:			
Bills	3,696,169	1,217	3,697,386
Notes	13,745,309	6,608	13,751,917
Bonds	3,952,658	7,216	3,959,874
Treasury Inflation-Protected Securities	1,907,303	769	1,908,072
Floating Rate Notes [20]	617,196	8	617,204
Federal Financing Bank [1]	0	4,847	4,847
Total Marketable [a]	**23,918,635**	**20,665** [2]	**23,939,300**
Nonmarketable:			
Domestic Series	23,778	0	23,778
Foreign Series	264	0	264
State and Local Government Series	99,898	0	99,898
United States Savings Securities	173,500	0	173,500
Government Account Series	297,915	6,881,431	7,179,347
Other	3,603	0	3,603
Total Nonmarketable [b]	**598,958**	**6,881,431**	**7,480,390**
Total Public Debt Outstanding	**24,517,593**	**6,902,096**	**31,419,689**

Renegotiating loans

1.2.12 Smith has debts of 1000 due now and 1092 due two years from now. He proposes to repay them with a single payment of 2000 one year from now. What is the implied effective annual interest rate if the replacement payment is accepted as equivalent to the original debts?

What if two years from now what is due is 992 not 1092? (No such i and $i = 8.94\%$)

Nominal Interest Rates

The biannual report of CFPB on the consumer credit card market gives an average annual percentage rate (APR) of 20% for 2020.

What is this APR?

It is not an annual interest rate! The APR is an annual interest rate in **name only**, that's what nominal means.

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What is this APR?

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Example

A nominal annual interest rate of 24% compounded **monthly** actually means an interest rate of $\dfrac{24\%}{12} = 2\%$ monthly interest rate.

A nominal annual interest rate of 24% compounded **quarterly** means $\dfrac{24\%}{4} = 6\%$ quarterly interest rate.

Nominal Interest Rates

Definition

A **nominal annual interest rate** of j compounded m times a year means an actual interest rate $\dfrac{j}{m}$ compounded m times a year. The notation is $i^{(m)} = j$.

Example

Notation	Meaning
$i^{(12)} = 24\%$	$\frac{24\%}{12} = 2\%$ every $\frac{1}{12}$ of a year, i.e., every month
$i^{(6)} = 24\%$	$\frac{24\%}{6} = 4\%$ every $\frac{1}{6}$ of a year, i.e., every two months
$i^{(4)} = 24\%$	$\frac{24\%}{4} = 6\%$ every $\frac{1}{4}$ of a year, i.e., every quarter
$i^{(2)} = 24\%$	$\frac{24\%}{2} = 12\%$ every $\frac{1}{2}$ of a year, i.e., semiannually

Accumulation factor for Nominal vs Effective IR

Effective Annual IR from Nominal IR

Example

Say we borrow \$1000 on a credit card with 24% nominal interest rate. How much money do we owe after 1 year?

Effective Annual IR from Nominal IR

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Nominal IR	IR per period	Effective Annual IR	Debt
$i^{(1)}$	$j = 24\%$	$i = 24\%$	\$1240
$i^{(2)}$	$j = 12\%$	$i = 25.44\%$	\$1254.4
$i^{(4)}$	$j = 8\%$	$i = 26.248\%$	\$1262.48
$i^{(6)}$	$j = 4\%$	$i = 26.532\%$	\$1265.32
$i^{(12)}$	$j = 2\%$	$i = 26.824\%$	\$1268.24
$i^{(365)}$	$j = 0.0658\%$	$i = 27.115\%$	\$1271.15
$i^{(365 \cdot 24 \cdot 3600)}$	tiny per sec.	$i = 27.125\%$	\$1271.25

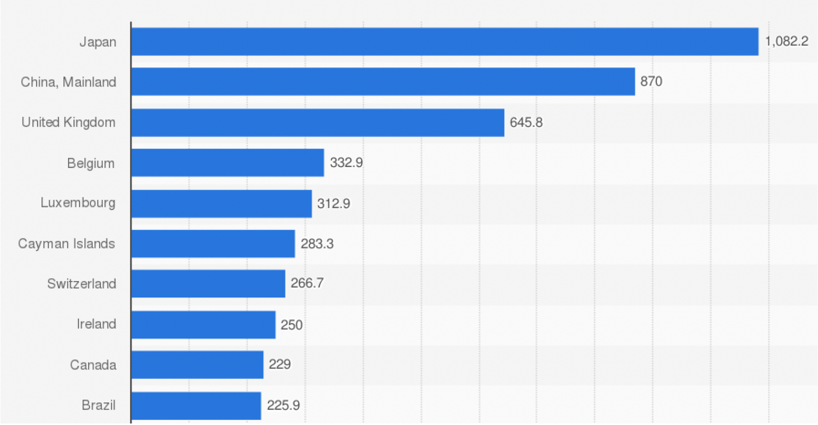
Nominal IR from Effective Annual IR

Example

A credit card company is aiming to issue a credit card with an effective annual IR of 24%. For accounting reasons it needs to compound interest daily and wants to quote a daily nominal interest rate. What should this APR be? (21.52%)

Lecture 8 February 3, 2023

US Public Debt Holders 11/2022



Effective Annual IR from Nominal IR: take 2

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Theorem

If the nominal interest rate $i^{(m)}$ has a fixed value, the effective annual interest rate grows as m grows.

Nominal IR: the number of periods

*1.4.10 Bank A has an effective annual rate of 18%. Bank B has a nominal annual rate of 17%. What is the smallest whole number of times per year that Bank B must compound its interest in order that the rate at Bank B be at least as attractive as that at Bank A on an effective annual basis? Repeat the exercise with a nominal rate of 16% at Bank B.

Nominal IR: the number of periods

*1.4.10 Bank A has an effective annual rate of 18%. Bank B has a nominal annual rate of 17%. What is the smallest whole number of times per year that Bank B must compound its interest in order that the rate at Bank B be at least as attractive as that at Bank A on an effective annual basis? Repeat the exercise with a nominal rate of 16% at Bank B.

Nominal	Effective	Nominal	Effective
$i^{(1)} = 17\%$	$i = 17\%$	$i^{(1)} = 16\%$	$i = 16\%$
$i^{(2)} = 17\%$	$i = 17.98\%$	$i^{(2)} = 16\%$	$i = 16.64\%$
$i^{(3)} = 17\%$	$i = 18.11\%$	$i^{(3)} = 16\%$	$i = 16.87\%$
$i^{(4)} = 17\%$	$i > 18.11\%$	$i^{(4)} = 16\%$	$i = 16.99\%$
$i^{(5)} = 17\%$	$i > 18.11\%$	$i^{(5)} = 16\%$	$i = 17.06\%$
$i^{(6)} = 17\%$	$i > 18.11\%$	$i^{(6)} = 16\%$	$i = 17.11\%$
		$i^{(12)} = 16\%$	$i = 17.23\%$
		$i^{(24)} = 16\%$	$i = 17.29\%$
		$i^{(365)} = 16\%$	$i = 17.347\%$
		$i^{(365\cdot 24\cdot 3600)} = 16\%$	$i = 17.351\%$

Nominal IR compounded "infinitely often"

Theorem

If the nominal interest rate $i^{(m)}$ has a fixed value, the effective annual interest rate grows as m grows.

Nominal IR compounded "infinitely often"

Theorem

If the nominal interest rate $i^{(m)}$ has a fixed value, the effective annual interest rate grows as m grows.

Example

In 1.4.10 if the nominal interest rate were 16%, even if we compounded trillions times per second we could never get an effective annual interest rate above

$$i_{\text{Effective}} = e^{0.16} - 1 = 17.351087\%$$

Vice-versa, if we aimed for an effective annual IR of 18% we could never have a nominal IR smaller than

$$i^{(\infty)}_{\text{Nominal}} = \ln(1 + 0.18) = 16.5514438477\%$$

Discount Rate

Careful

There are two different notions of **discount rate** in finance. The more common discount rate is the interest charged by the Federal Reserve to banks for overnight loans. This is unrelated to the discount rate in this course, the textbook, and actuary science.

Example

Say you purchase a 28-day Treasury Bill, which pays $100 in 28 days, for $95. How do you measure how well your investment is doing?

Discount Rate

Example

Say you purchase a 28-day Treasury Bill, which pays \$100 in 28 days, for \$95. How do you measure how well your investment is doing?

- Yield rate: we made a \$5 profit on a \$95 investment so

$$i = \frac{100 - 95}{95} = 5.26\%.$$

Discount Rate

Example

Say you purchase a 28-day Treasury Bill, which pays \$100 in 28 days, for \$95. How do you measure how well your investment is doing?

- Yield rate: we made a \$5 profit on a \$95 investment so

$$i = \frac{100 - 95}{95} = 5.26\%.$$

- Discount rate: we bought a \$100 bill for only \$95 so at a "discount" of

$$d = \frac{100 - 95}{100} = 5\%.$$

Discount Rate

Definition

The **discount rate** of a financial asset or investment is

$$d = \frac{A(1) - A(0)}{A(1)}.$$

The yield rate, or effective interest rate, which is

$$i = \frac{A(1) - A(0)}{A(0)},$$

and the present value factor is

$$\nu = \frac{A(0)}{A(1)}.$$

Relations between interest, discount, present value factor

	i	ν	d
i	$-$	$\frac{1}{\nu} - 1$	$\frac{d}{1-d}$
ν	$\frac{1}{1+i}$	$-$	$1-d$
d	$\frac{i}{1+i}$	$1-\nu$	$-$

Discount across time

Example

Suppose my investment earns at a discount rate of 5% for 3 years in a row. What is the total discount for the entire 3-year period? (14.26%)

Simple interest vs simple discount

After time t we have:

Concept	With compound interest
Accumulation factor at time t	$(1+i)^t$
Present value of \$1 at time t	$\nu^t = (1-d)^t$

Terminology:

- **Simple interest** means that the accumulation factor at time t is $a(t) = 1 + it$.
- **Simple discount** means that the present value of \$1 at time t is $1 - dt$.

Lecture 9 February 6, 2023

Feb 6, 5:39:59 PM UTC · Disclaimer

Bitcoin (BTC / USD)	23,054.60	-19,360.50	↓45.65%
Ether (ETH / USD)	1,650.02	-1,408.20	↓46.05%
Binance Coin (BNB ...	328.4527	-90.0974	↓21.53%

Recall from last lecture

	i	ν	d
i	$-$	$\frac{1}{\nu} - 1$	$\frac{d}{1-d}$
ν	$\frac{1}{1+i}$	$-$	$1-d$
d	$\frac{i}{1+i}$	$1-\nu$	$-$

- **Simple interest** means that the accumulation factor at time t is $a(t) = 1 + it$.
- **Simple discount** means that the present value of \$1 at time t is $1 - dt$.

Example for simple discount: T-Bills

Example

We buy a 28-day Treasury Bill for \$99.

1. What is the effective annual discount? **Note:** Treasury Bills use a 360-day year and simple discount. (12.86%)
2. Compare to the answer you'd get with compound interest, and simple interest. $(12.12\%$ resp. $11.49\%)$

T-Bills today

	4 WEEKS		8 WEEKS		13 WEEKS		17 WEEKS	
Date	BANK DISCOUNT	COUPON EQUIVALENT	BANK DISCOUNT	COUPON EQUIVALENT	BANK DISCOUNT	COUPON EQUIVALENT	BANK DISCOUNT	CO EQ
01/03/2023	3.96	4.03	4.29	4.38	4.40	4.51	4.57	4.7
01/04/2023	4.00	4.07	4.28	4.37	4.41	4.52	4.58	4.7
01/05/2023	4.12	4.19	4.44	4.53	4.51	4.62	4.62	4.7
01/06/2023	4.14	4.21	4.43	4.52	4.51	4.62	4.61	4.7
01/09/2023	4.17	4.24	4.44	4.53	4.55	4.67	4.60	4.7
01/10/2023	4.20	4.27	4.47	4.56	4.57	4.69	4.64	4.7
01/11/2023	4.20	4.27	4.48	4.57	4.57	4.69	4.69	4.8
01/12/2023	4.37	4.45	4.47	4.56	4.50	4.61	4.61	4.7
01/13/2023	4.38	4.46	4.46	4.55	4.51	4.62	4.60	4.7
01/17/2023	4.40	4.48	4.50	4.59	4.56	4.68	4.61	4.7
01/18/2023	4.39	4.47	4.49	4.58	4.54	4.66	4.61	4.7
01/19/2023	4.49	4.57	4.53	4.63	4.55	4.67	4.61	4.7
01/20/2023	4.48	4.56	4.51	4.60	4.57	4.69	4.61	4.7
01/23/2023	4.49	4.57	4.52	4.62	4.58	4.70	4.62	4.7

Discount rate: an exercise

1.5.5S Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits 100 into his bank account, and Robbie deposits 50 into his. Each account earns an effective annual discount rate of d. The amount of interest earned in Bruce's account during the 11^{th} year is equal to X. The amount of interest earned in Robbie's account during the 17^{th} year is also equal to X. Calculate X.

$(d = 10.91\%,\ X = 38.89)$

Nominal discount rate

Definition

We say that $d^{(m)}$ is the **nominal discount rate** compounded m times per year if the actual discount rate is $\frac{d^{(m)}}{m}$ per each period of $\frac{1}{m}$ years.

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$$1 - d = \nu_{\text{per year}} = \nu_{\text{per period}}^{m} = \left(1 - \frac{d^{(m)}}{m}\right)^{m}$$

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$$1 + i = a(1) = a\left(\frac{1}{m}\right)^{m} = \left(1 + \frac{i^{(m)}}{m}\right)^{m}$$

Lecture 10 February 8, 2023

Facebook stock this week

Recall from last lecture

Nominal Interest and Discount Rates

Nominal discount rate: an exercise

1.5.7S Jeff deposits 10 into a fund today and 20 fifteen years later. Interest is credited at a nominal discount rate of d compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semiannually thereafter. The accumulated balance in the fund at the end of 30 years is 100. Calculate d.

(4.53%)

Measuring yield rates over short periods

(223%, 260%, 11%: annualized to 6069%, 413%, 23%)

Instantaneous Effective Annual Yield Rate

Suppose you wanted to compute the yield rate over a short period of time, say from $t-h$ to t. The actual yield rate $i(t-h,t)$ between $t-h$ and t is given by

$$1+i(t-h,t)=\frac{A(t)}{A(t-h)}.$$

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To compare, we need to find an effective annual measure

$$1 + i_{\text{Effective}}(t-h,t) = (1 + i(t-h,t))^{1/h} = \left(\frac{A(t)}{A(t-h)}\right)^{\frac{1}{h}}.$$

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An instantaneous effective annual yield rate would then be given by

$$1 + i_{\text{Effective},t} = \lim_{h\to 0} 1 + i_{\text{Effective}}(t-h,t) = \lim_{h\to 0}\left(\frac{A(t)}{A(t-h)}\right)^{\frac{1}{h}}$$

Instantaneous Yield Rate: calculus

What is

$$1 + i_{\text{Effective},t} = \lim_{h \to 0} 1 + i_{\text{Effective}}(t-h,t) = \lim_{h \to 0} \left(\frac{A(t)}{A(t-h)} \right)^{\frac{1}{h}}$$

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$$\ln(1+i_{\text{Effective},t}) = \lim_{h \to 0} \ln \left(\frac{A(t)}{A(t-h)} \right)^{\frac{1}{h}} = \lim_{h \to 0} \frac{1}{h} \ln \left(\frac{A(t)}{A(t-h)} \right)$$

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What is

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$$\ln(1 + i_{\text{Effective},t}) = \frac{A'(t)}{A(t)}.$$

Force of Interest

What does the expression $\ln(1 + i_{\text{Effective},t})$ remind you of?

Force of Interest

What does the expression $\ln(1 + i_{\text{Effective},t})$ remind you of? Indeed,

$$i^{(\infty)}_{\text{Effective},t} = \ln(1 + i_{\text{Effective},t})$$

is the corresponding nominal annual interest rate compounded continuously.

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is the corresponding nominal annual interest rate compounded continuously.

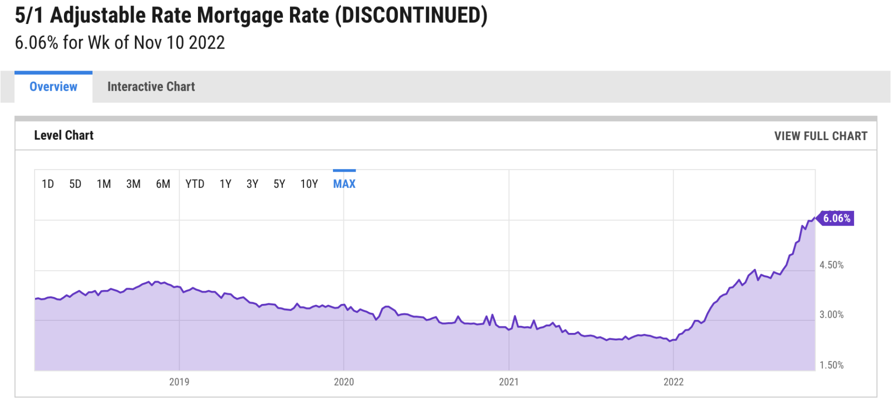
Force of Interest

The **force of interest** of an investment is the time-varying function

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt} \ln A(t) = \frac{d}{dt} \ln a(t)$$

equal to the **nominal annual interest rate compounded continuously** that corresponds to the **instantaneous yield rate** measured at time t.

Lecture 11 February 10, 2023



Recall from last lecture

Force of Interest

Force of Interest: Examples

Example

Compount interest i: $a(t) = (1+i)^t$ so

$$\delta_t = \frac{d}{dt}\ln((1+i)^t) = \frac{d}{dt}t\ln(1+i) = \ln(1+i) = i^{(\infty)}.$$

Force of Interest: Examples

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Nominal interest rate $i^{(m)}$: $a(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{mt}$ so

$$\delta_t = \frac{d}{dt}\ln\left(\left(1+\frac{i^{(m)}}{m}\right)^{mt}\right) = \frac{d}{dt}mt\ln\left(1+\frac{i^{(m)}}{m}\right)$$

$$= m\ln\left(1+\frac{i^{(m)}}{m}\right).$$

Force of Interest: Examples

Example

Discount rate d: $a(t) = \left(\frac{1}{\nu}\right)^t = \left(\frac{1}{1-d}\right)^t$ so

$$\delta_t = \frac{d}{dt} \ln \left(\frac{1}{1-d}\right)^t = \frac{d}{dt} t \ln \left(\frac{1}{1-d}\right) = -\ln(1-d).$$

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Simple interest i: $a(t) = 1 + it$ so

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Force of Interest: Examples

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Simple interest i: $a(t) = 1 + it$ so

$$\delta_t = \frac{d}{dt}\ln(1+it) = \frac{i}{1+it}$$

Simple discount d: $\mathrm{PV} = 1 - dt$ so $a(t) = \frac{1}{1-dt}$

$$\delta_t = \frac{d}{dt}\ln\left(\frac{1}{1-dt}\right) = \frac{d}{dt}\left(-\ln\left(1-dt\right)\right) = \frac{d}{1-dt}.$$

Force of Interest: Exercise

1.6.2

1.6.2S A customer is offered an investment where interest is calculated according to the following force of interest:

$$\delta_t = \begin{cases} .02t & 0 \le t \le 3 \\ .045 & t > 3 \end{cases}$$

The customer invests 1000 at time $t = 0$. What nominal rate of interest, compounded quarterly, is earned over the first four-year period?

$(i^{(4)} = 3.39\%)$

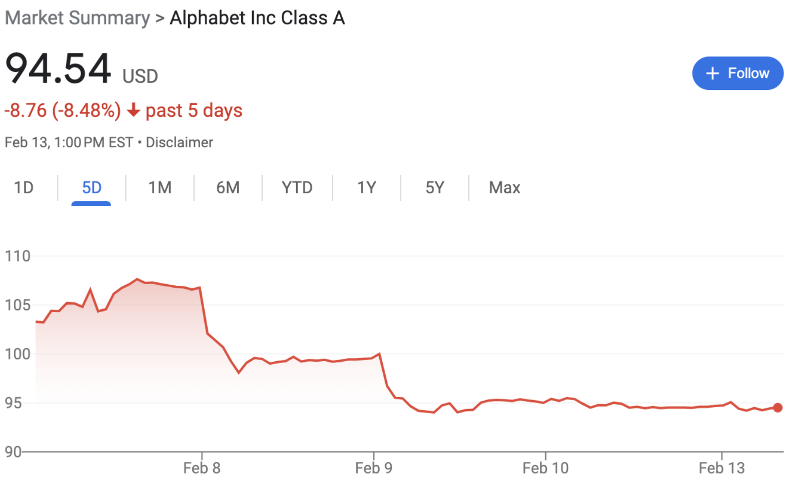
Force of Interest: Exercise

1.6.5

1.6.5S Ernie makes deposits of 100 at time 0, and X at time 3. The fund grows at a force of interest $\delta_t = \frac{t^2}{100}$, $t > 0$. The amount of interest earned from time 3 to time 6 is X. Calculate X.

(784.6)

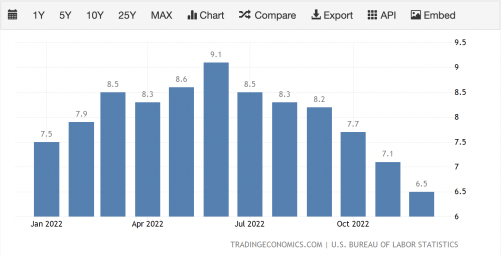
Lecture 12 February 13, 2023



How Inflation Affects Interest/Yields

Inflation

In 2022 **inflation** was 6.5%, food inflation 10.5%, and inflation excepting food and energy was 5.7%. For instance, this means your food budget should have increased by 10.5% in 2022 just to be able to afford the same food as a year before. For comparison, the numbers in 2021 were 7%, 6.3%, and 5.5%.



How Inflation Affects Interest/Yields

Example

In 2022 the stock market had the following yields.

Index	2022 Return	2021 Return
S&P 500	-19.44%	26.9%
Dow Jones	-8.74%	18.7%
NASDAQ	-33.47%	21.4%

What would be the yield rate adjusting for inflation?

How Inflation Affects Interest/Yields

Example

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Index	2022 Return	2021 Return
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What would be the yield rate adjusting for inflation?

Index	2022	2021 Inflation-Adjusted Return
S&P 500	-24.36%	18.6%
Dow Jones	-14.31%	10.9%
NASDAQ	-37.53%	13.5%

Inflation-Adjusted Yield Rate

Suppose your investment has a yield rate/effective annual interest rate of i, and the effective annual inflation rate over the same period is r. What is your actual, or **inflation-adjusted** yield rate?

Inflation-Adjusted Yield Rate

$$\text{Inflation-Adjusted Yield Rate} = \frac{i - r}{1 + r}.$$

Yield Rate if you borrow

Example

Suppose you see the following investment opportunity over the course of a month:

- you can borrow at 3.25% effective annual and
- you can invest at 5.5% effective annual.

What is your effective annual yield relative to how much you stand to lose? (2.18%)

Yield Rate Across Currencies

Example

Suppose, now, that you have \$1 million. You decide on January 1st, 2022 to make an investment over the next month, but all USD investments seem to have small yield rates. You do, however, find a Romanian financial asset that pays 5.5% effective annual.

Yield Rate Across Currencies

Example

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If you buy RON, invest at 5.5%, then reconvert into USD, what is your effective annual yield rate? The exchange rate for USDRON is 4.35135 on 1/1 and 4.39332 on 2/1. ($r = 0.96\%$ and yield rate is 4.49%)

What's next?

- Annuities §2
- Loans §3
- Bonds §4
- How well is an investment performing? §5
- How do interest rates vary across term? §6
- Interest rate hedging §7