Annuities

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Example

- Monthly deposits of $1000 into an interest-earning account.

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- Paying a lump sum at retirement and receiving monthly pension payments for the rest of your life.

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- Lottery winnings payable over 30 years.

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- Pension fund deposits.
- Paying a lump sum at retirement and receiving monthly pension payments for the rest of your life.
- Lottery winnings payable over 30 years.
- Mortgages and other such loans.

Annuities: Basic Setup

The most basic annuity has the following assumptions:

- Constant payments of C at
- n equally spaced intervals
- and constant interest rate i.

Annuities: Basic Setup

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Question

How much is an annuity worth at various times? For example:

- *If I can afford monthly payments of $400 for the next 5 years, how much money can I borrow right now to buy a car?*
- *What should monthly payments be on my mortgage?*
- *If I make monthly deposits of $100 into an education fund, how much money will be in the account when my child starts college?*

Annuities: Basic Setup

The most basic annuity has the following assumptions:

- n constant payments of C at
- equally spaced intervals
- and constant interest rate i.

Definition

The concise notation for n payments at equally spaced periods and constant interest rate i (per period) is $\overline{n|}i$ written in the subscript.

$$\text{Annuity}_{\overline{n|}i}$$

Two adjectives:

- Buy/sell an **annuity immediate** means the first periodic payment is at the end of the first period.
- Buy/sell an **annuity due** means the first periodic payment is at the start of the first period.

Lecture 13 February 15, 2023

Unemployment rate	Total unemployed people	Unemployment insurance claims
3.4%	5.7M	2.2M
Updated Jan 2023	Updated Jan 2023	Updated Feb 4, 2023

Unemployment rate | Total unemployed people | Unemployment insurance claims

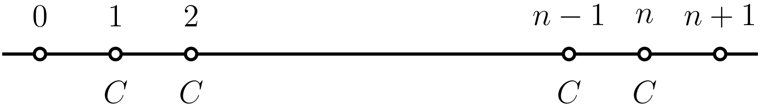
Unemployment rate is collected once a month · Numbers are seasonally adjusted

Recall from last lecture

Annuities

Annuity immediate and annuity due

What kind of contract is this?



Annuity immediate and annuity due

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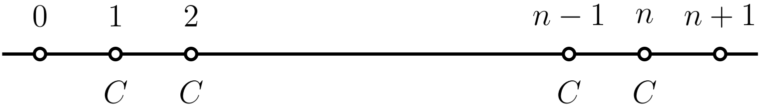


Contract length: n periods.

- Start at 0 end at n: annuity immediate.
- Start at 1 end at $n+1$: annuity due.

Example

For the following annuities, when will the first and last payments occur?

Purchase Date	Annuity Type	Frequency
4/1/2023	Annuity Immediate	Every 3 months
5/1/2023	Annuity Immediate	Yearly
6/1/2023	Annuity Due	Every month
7/1/2023	Annuity Due	Every 6 months

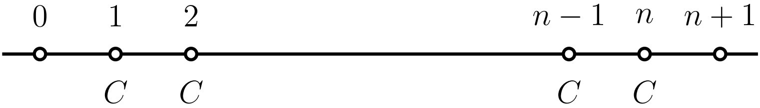
Annuity immediate and annuity due

Example

At the start of 2023 you buy an annuity immediate with 10 yearly payments of 100, you sign a contract to buy at the start of 2025 an annuity immediate with 5 payments of 200 every two years, and another contract to sell an annuity due with 4 payments of 150 every 3 years. What will be the total cashflows at the start of 2029, 2030, and 2031?

Present Value and Accumulated Value

What kind of contract is this?



Time value of annuity

What is the value of this annuity $\text{Annuity}_{\overline{n}|i}$ at various points in time?

Time t	Notation for value of $\text{Annuity}_{\overline{n}|i}$ at time t
0	$a_{\overline{n}|i}$
1	$\ddot{a}_{\overline{n}|i}$
n	$s_{\overline{n}|i}$
$n+1$	$\ddot{s}_{\overline{n}|i}$

Pricing an annuity at the end: accumulated value

Pricing an annuity at the end: algebra

$$s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1}$$

$$1 + x + x^2 + \cdots + x^k = \frac{x^{k+1} - 1}{x - 1}$$

Pricing an annuity at the end: algebra

$$s_{\overline{n}|i} = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1}$$

$$1 + x + x^2 + \cdots + x^k = \frac{x^{k+1} - 1}{x - 1}$$

$$s_{\overline{n}|i} = \frac{(1+i)^{n-1+1} - 1}{(1+i) - 1}$$

Accumulated Value of Annuity Immediate

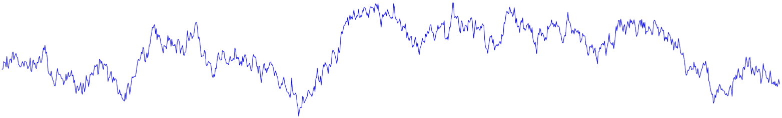
$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

Lottery winnings

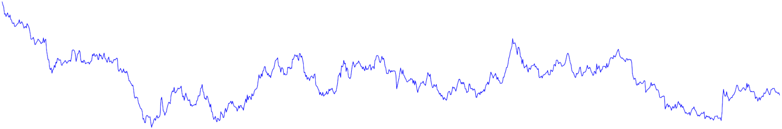
Example

You won the lottery and you received $1000 each year for 30 years. Interest rates are 3%. What is the value of the lottery winnings at the time of the last payment? What about 10 years later?

Lecture 14 February 17, 2023



One of these two plots is EURUSD in 2015. The other one is the time series that pays $\pm\$1$ for each prime number p with $p \pm 1$ a multiple of 4. Which one is which?



Recall from last lecture

Annuities

Annuities after the last payment

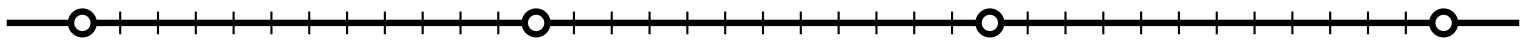
Accumulated value after the last payment

To find the accumulated value of an annuity after the last payment you simply multiply by the accumulated factor, as you would do for any balance.

Annuities after the last payment

Example

On the first of each month of 2022 you deposit \$1000, on the first of each month of 2023 you deposit \$2000, and on the first of each month of 2024 you deposit \$3000. How much money do you have at the end of 2024 if the nominal interest rate compounded monthly was 5% in 2022, 4% in 2023, and 3% in 2024? (75085.47)



Pricing an annuity at the beginning

Pricing an annuity at the beginning: algebra

$$a_{\overline{n}|i} = \nu + \nu^2 + \cdots + \nu^n$$

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}$$

Pricing an annuity at the beginning: algebra

$$a_{\overline{n}|i} = \nu + \nu^2 + \cdots + \nu^n$$

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}$$

$$a_{\overline{n}|i} = \nu \cdot \frac{1 - \nu^n}{1 - \nu}$$

Present Value of Annuity Immediate

$$a_{\overline{n}|i} = \frac{1 - \nu^n}{i}$$

Pricing annuities: finding the present value

Example

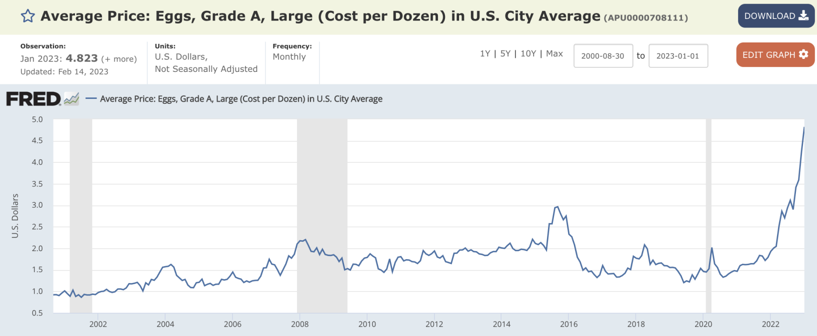
I want to buy a car, and the bank offers me a loan at nominal interest rate $i^{(12)} = 2.3\%$ for a 5-year loan. I can afford to pay 300 per month. How much money can I borrow to buy the car? (16988.2. In 2021 the average price for used cars was $\$28205$.)

Pricing annuities: finding the payments

Example

On $2/7/2022$ a 30-year fixed mortgage sold near Notre Dame with a monthly nominal annual interest rate of 3.7%. If you borrowed \$260000 for a house, what are your monthly payments? (1196.74)

Lecture 15 February 20, 2023



Pricing an annuity at the beginning: algebra take 2

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$
$$a_{\overline{n}|i} = \frac{1 - \nu^n}{i}$$

Pricing annuities: finding the interest rate

2.1.17

2.1.17S At an effective annual interest rate of $i, i > 0$, both of the following annuities have a present value of X:

(a) a 20-year annuity-immediate with annual payments of 55

(b) a 30-year annuity-immediate with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years.

Calculate X.

($i = 7.177\%$ and $X = 574.74$)

Annuities at other times

What do you need to remember?

If you know the price of an annuity at **any** time you can find its value at **any other** time, using the concept of time value of money.

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \qquad a_{\overline{n}|i} = \frac{1-\nu^n}{i}$$

$$\ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d} \qquad \ddot{a}_{\overline{n}|i} = \frac{1-\nu^n}{d}$$

Pricing annuities: finding the number of periods

Example

\$10k are borrowed at the present, to be repaid in yearly installments, beginning with one year from now. The payments are \$500 per year for the first 5 years, and \$1000 per year thereafter. The annual interest rate is 4%. How long until the debt is repaid? (18 years)

Combining annuities: different levels of payments

2.1.31

*2.1.31 Smith borrows 5000 on January 1, 2005. She repays the loan with 20 annual payments starting January 1, 2006. The payments in even-numbered years are Y each and the payments in odd-numbered years are X each. If $i = .08$ and the total of all 20 loan payments is 10,233, find X and Y.

($X = 573.36$ and $Y = 449.94$)

Lecture 16 February 22, 2023

Midterm 1: Chapters 1 and 2.1

Lecture 17 February 24, 2023

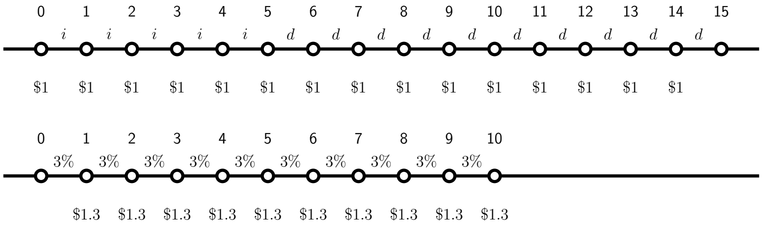
Recall from last lecture

Annuities

Combining annuities: different interest rates

Example

A 15-year annuity has yearly payments starting now. For the first 5 years the annual interest rate is 4%, and for the last 10 years it has annual discount rate of d. You make level payments. What is the implied discount rate d if this annuity is equivalent to a 10-year annuity immediate with interest rate 3% and level payment rates that are 30% larger than in the first annuity? $(d = 5.49\%)$

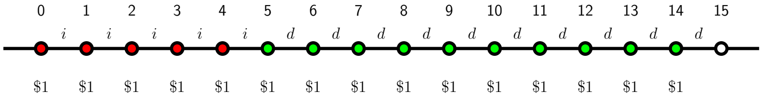


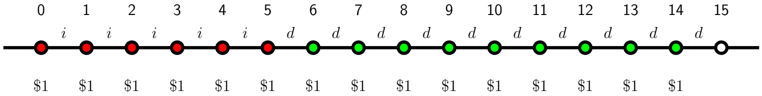
Combining annuities: different interest rates

Basic Annuity

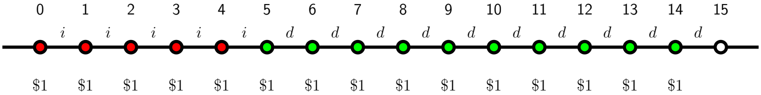
Remember that a basic annuity (for which we can use $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$) has 2 assumptions: (a) equal payments, and (b) equal interest rate.

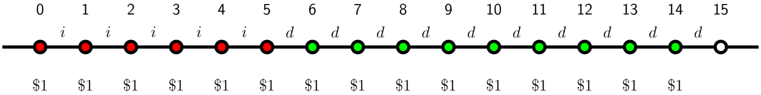
The first annuity can be realized as a combination of two basic annuities in two different ways:





Combining annuities: different interest rates





Forever annuities

Perpetuity

A **perpetuity** is an annuity with infinitely many equally spaces level payments at a fixed interest rate.

Remark

- *Perpetuities are, really, financial fictions. They were issued in the past, under the name of "consols," by the UK (until 2015) and the US (until 1930).*
- *But they are useful mathematical devices that can be used to value certain financial assets, such as stock prices, real estate assets, etc.*

Pricing Perpetuities

$$\text{Perpetuity}_i = \text{Annuity}_{\overline{\infty}|i}$$

Pricing Perpetuities

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$$a_{\overline{\infty}|i} = \lim_{n\to\infty} a_{\overline{n}|i} = \lim_{n\to\infty} \frac{1-\nu^n}{i} = \frac{1}{i}$$

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Alternatively, we could use the timeline again:

$$a_{\overline{\infty}|i} = \nu + \nu^2 + \nu^3 + \cdots = \nu(1 + \nu + \nu^2 + \cdots) = \frac{\nu}{1-\nu} = \frac{1}{i}.$$

Pricing Perpetuities

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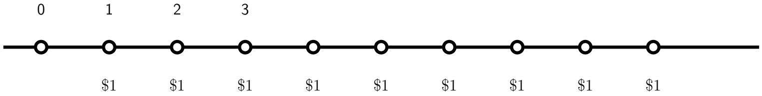
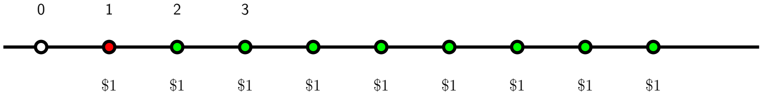
Remark

This is related (and mathematically identical) to questions about probability: You throw a pair of dice. How long until you hit a double 6? The probability to get a double 6 is $i = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6}$. *The expected number of tries until you get a double 6 is then* $\frac{1}{i} = 36$.

Pricing Perpetuities: take 2



Write $P = a_{\overline{\infty}|i}$, the price of a perpetuity one period before the first payment.



How close are perpetuities to reality?

Example

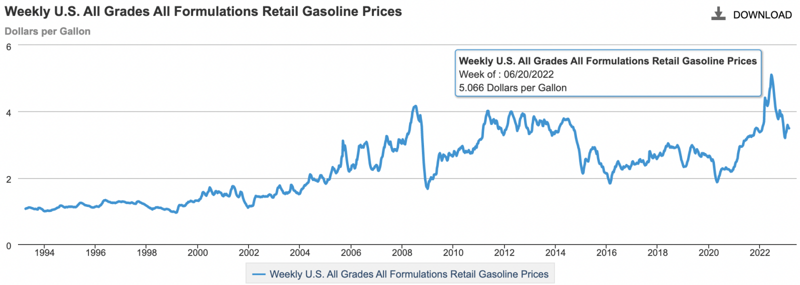
Pride and Prejudice features 4% perpetuities, and such assets were widely purchased and traded in Jane Austen's time. (Also World War 2 Britain.) Such a perpetuity purchased in 1813 (the year the novel appeared) would have been redeemed by the UK Parliament in 1923.

How close are perpetuities to reality?

Example

Pride and Prejudice features 4% perpetuities, and such assets were widely purchased and traded in Jane Austen's time. (Also World War 2 Britain.) Such a perpetuity purchased in 1813 (the year the novel appeared) would have been redeemed by the UK Parliament in 1923. What if in 1923 this perpetuities had been lost (for example, the UK goverment might have been bankrupt after World War 1). How close would the value of this financial asset (a 110-year annuity) have come to the theoretical value of a perpetuity? (Within 1.34%, within 1% after 118 years.)

Lecture 18 February 27, 2023



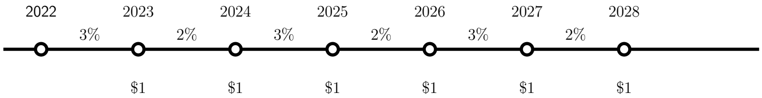
Recall from last lecture

Annuities and Perpetuities

Combining perpetuities

Example

A perpetuity pays 1 every January 1 starting in 2023. The effective annual interest rate will be 2% in odd-numbered years and 3% in even-numbered years. Find the present value of the perpetuity on January 1, 2022. (39.92)



Pricing perpetuities before they begin

Example

A sum of \$100k is donated to Notre Dame on September 1, 2022, to be added to the endowment in order to finance the Math Club. Notre Dame invests this sum at an effective annual interest rate of 5%, in order to provide the Math Club \$10k every September 1 forever, starting as soon as possible. In what year with the first payment of \$10k be made? (2038, but it could afford to also pay \$7892.82 in 2036.)

Recovering annuities from perpetuities

We already saw that

$$\text{Annuity}_{\overline{n}|i}(1,\ldots,n) = \text{Annuity}_{\overline{\infty}|i}(1,\ldots) - \text{Annuity}_{\overline{\infty}|i}(n+1,\ldots)$$

$$\text{PV}_{t=0}(\text{Annuity}_{\overline{n}|i}(1,\ldots,n)) =$$
$$\text{PV}_{t=0}(\text{Annuity}_{\overline{\infty}|i}(1,\ldots)) =$$
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Recovering annuities from perpetuities

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$$\text{PV}_{t=0}(\text{Annuity}_{\overline{\infty}|i}(n+1,\ldots)) =$$

$$a_{\overline{n}|i} = a_{\overline{\infty}|i} - \nu^n a_{\overline{\infty}|i} = \frac{1-\nu^n}{i}.$$

$$s_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n = \frac{(1+i)^n - 1}{i}.$$

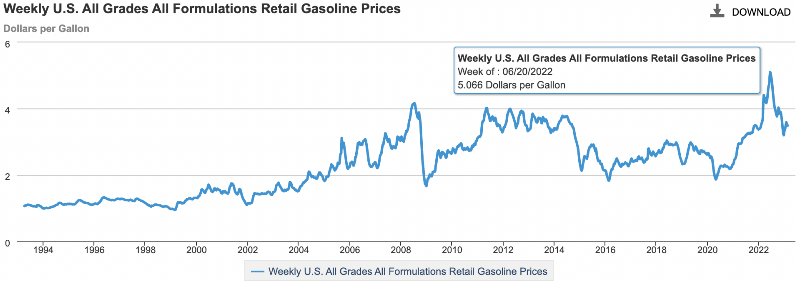
In-class Exercise

Example

You purchase a perpetuity immediate on $1/1/2022$ for X. This perpetuity pays every $1/1$ as follows: if the year is a multiple of 3, it pays 3; if the year is a multiple of 3 plus 1, it pays 5; if the year is a multiple of 3 plus 2, it pays 7.

- What was the price X of this annuity if the interest rate is 10%?
- What about if interest rates are 5% in even years, and 10% in odd years?

Lecture 19 March 1, 2023



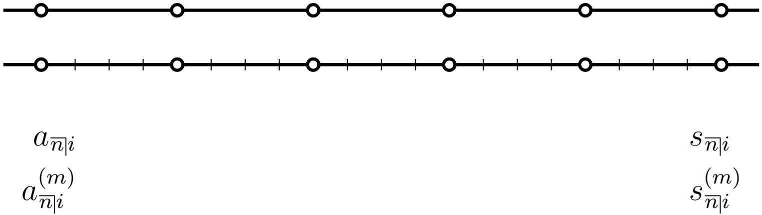
Annuities payable multiple times per period

m-thly payable annuity

An m-**thly payable annuity** is an annuity lasting n years and paying $\frac{1}{m}$ each period of $\frac{1}{m}$ years.

Remark

An n-year annuity and an n-year m-thly payable annuity have the same time length and the same total cash paid, but of course the values of these two cashflows will be different.



Pricing m-thly payable annuities

Pricing m-thly payable annuities

How did we approach questions like this previously? We computed the per-period interest j and reinterpreted the n-year annuity as an mn-period annuity:

$$\text{Annuity}^{(m)}_{\overline{n}|i} = \frac{1}{m}\,\text{Annuity}_{\overline{mn}|j}\,.$$

You could do

$$a^{(m)}_{\overline{n}|i} = \frac{1}{m}a_{\overline{mn}|j} \qquad s^{(m)}_{\overline{n}|i} = \frac{1}{m}s_{\overline{mn}|j}$$

Pricing m-thly payable perpetuities

Instead, let's work out m-thly payable perpetuities:

$$\text{Annuity}^{(m)}_{\overline{\infty|}i} = \frac{1}{m}\,\text{Annuity}_{\overline{\infty|}j}$$

$$a^{(m)}_{\overline{\infty|}i} = \frac{1}{m} a_{\overline{\infty|}j}$$

$$1 + i = (1 + j)^m = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

Pricing m-thly payable perpetuities

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$$a^{(m)}_{\overline{\infty}|i} = \frac{1}{m} a_{\overline{\infty}|j}$$

$$1 + i = (1+j)^m = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

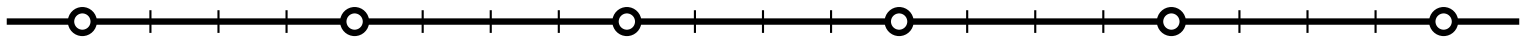
$$a^{(m)}_{\overline{\infty}|i} = a_{\overline{\infty}|i} \times \frac{i}{i^{(m)}}$$

$$\text{Annuity}^{(m)}_{\overline{\infty}|i} = \text{Annuity}_{\overline{\infty}|i} \times \frac{i}{i^{(m)}}$$

Pricing annuities vs m-thly payable annuities

$$\text{Annuity}_{\overline{n}|i} = \text{Annuity}_{\overline{\infty}|i}^{\text{year}>0} - \text{Annuity}_{\overline{\infty}|i}^{\text{year}>n}$$

$$\text{Annuity}_{\overline{n}|i}^{(m)} = \text{Annuity}_{\overline{\infty}|i}^{(m),\text{year}>0} - \text{Annuity}_{\overline{\infty}|i}^{(m),\text{year}>n}$$



so we conclude that

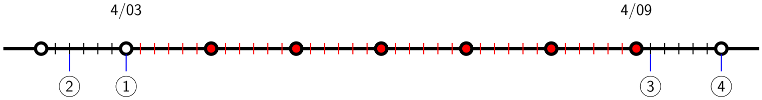
$$\text{Annuity}_{\overline{n}|i}^{(m)} = \text{Annuity}_{\overline{n}|i} \times \frac{i}{i^{(m)}}$$

$$a_{\overline{n}|i}^{(m)} = a_{\overline{n}|i} \times \frac{i}{i^{(m)}} \qquad s_{\overline{n}|i}^{(m)} = s_{\overline{n}|i} \times \frac{i}{i^{(m)}}$$

Exercise

2.2.16 Payments of 25 are made every 2 months from June 1, 2003 to April 1, 2009 inclusive. Find the value of the series

1. 2 months before the first payment at an annual effective interest rate of $i = 0.06$ (755.83),
2. 10 months before the first payment at a nominal annual rate of $i^{(3)} = 0.06$ (724.08),
3. 2 months after the final payment at a nominal annual rate $d^{(2)} = 0.06$ (1092.02), and
4. 1 year after the final payment at annual force of interest $\delta = 0.06$ (1144.57).



Exercise

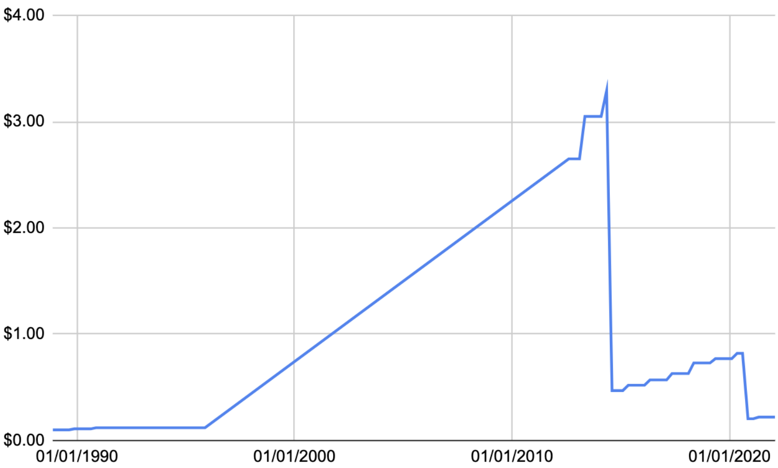
2.2.15

At an effective annual interest rate of $i, i > 0$, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 6, is 32 . At the same effective annual rate of i, the present value of a perpetuity-immediate paying 1 at the end of each 4-month period is X. Calculate X.

($i = 7.72\%$ and $X = 39.83$)

Lecture 20 March 3, 2023

Apple Dividends



Recall from last lecture

Annuities and Perpetuities

Continuously paying annuities

Recall from the chapter on nominal interest rates the idea of a continuous nominal interest rate $i^{(\infty)}$, which we reinterpretted as **force of interest**.

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m = e^{i^{(\infty)}} = e^{\delta}$$

$$\delta = i^{(\infty)} = \ln(1 + i)$$

$$a(t) = (1 + i)^t = e^{\delta t}$$

Continuous Annuities

A continuous annuity is an annuity of the form $\text{Annuity}^{(m)}_{\overline{n}|i}$ as $m \to \infty$. It pays continuously for a total of 1 per period.

Pricing continuous annuities

The present value and accumulated value of a continuously paying annuity are

$$\bar{a}_{\overline{n}|i} = \text{“}a^{(\infty)}_{\overline{n}|i}\text{”} = \lim_{m\to\infty} a^{(m)}_{\overline{n}|i}$$

$$\bar{s}_{\overline{n}|i} = \text{“}s^{(\infty)}_{\overline{n}|i}\text{”} = \lim_{m\to\infty} s^{(m)}_{\overline{n}|i}$$

Pricing continuous annuities

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$$\bar{s}_{\overline{n}|i} = \text{“}s^{(\infty)}_{\overline{n}|i}\text{”} = \lim_{m\to\infty} s^{(m)}_{\overline{n}|i}$$

In other words

$$\bar{a}_{\overline{n}|i} = a_{\overline{n}|i}\frac{i}{\delta}$$

$$\bar{s}_{\overline{n}|i} = s_{\overline{n}|i}\frac{i}{\delta}$$

Pricing continuous annuities with force of interest

How to price continuous annuities if the interest rate is not constant? I.e., with variable force of interest?

$$\bar{a}_{\overline{n}|i} = \int_0^n \frac{1}{a(0,t)} dt \qquad \bar{s}_{\overline{n}|i} = \int_0^n a(t,n) dt$$

Pricing continuous annuities with force of interest

How to price continuous annuities if the interest rate is not constant? I.e., with variable force of interest?

$$\bar{a}_{\overline{n}|i} = \int_0^n \frac{1}{a(0,t)} dt = \int_0^n (1+i)^{-t} dt$$

$$\bar{s}_{\overline{n}|i} = \int_0^n a(t,n) dt = \int_0^n (1+i)^{n-t} dt$$

Pricing continuous annuities with force of interest

How to price continuous annuities if the interest rate is not constant? I.e., with variable force of interest?

$$\bar{a}_{\overline{n}|i} = \int_0^n \frac{1}{a(0,t)}dt \qquad \bar{s}_{\overline{n}|i} = \int_0^n a(t,n)dt$$

$$= \int_0^n (1+i)^{-t}dt \qquad = \int_0^n (1+i)^{n-t}dt$$

With force of interest the only thing that changes is the accumulated factor: remembering that

$$\ln a(t_1,t_2) = \int_{t_1}^{t_2} \frac{d}{dt}\ln a(t)dt = \int_{t_1}^{t_2} \delta_t dt.$$

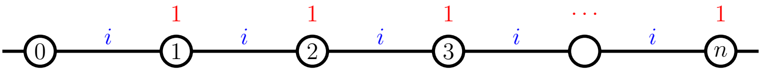
Exercise

2.2.18 Find the present value at time $t = 0$ of an 10-year continuous annuity based on force of interest

$$\delta_t = 0.1 + \frac{0.2}{1 + e^{0.2t}}.$$

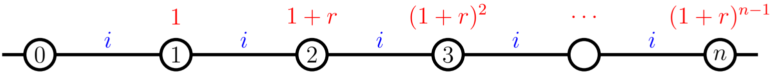
Annuities where the payments change over time

Our basic annuities had 3 assumptions: (a) equally spaced payments (b) constant interest rate i and (c) equal level of payments 1.



Three important alternatives keep assumptions (a) and (b), but relax (c):

- Geometric annuities with constant growth rate r

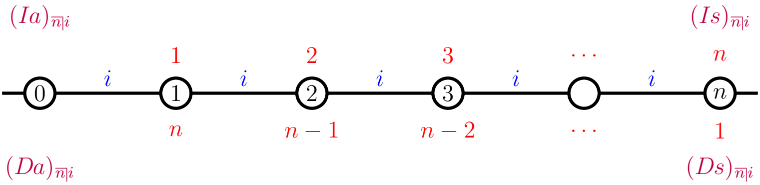


Example

The Powerball Lottery allows one to cash the winnings as a lump sum payable immediately OR as a 30-year geometric annuity due with $r = 5\%$.

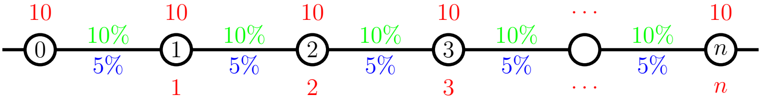
Annuities where the payments change over time

- Increasing/decreasing arithmetic annuities



Example

Say every year, starting now, you invest \$10 into an account which earns 10% but which does not cumulate interest. You withdraw the interest earn and reinvest it into an account at 5%.

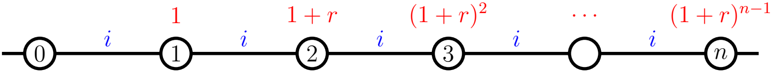


Pricing geometric annuities

Present value

Growth rate r, interest rate i, n periods:

$$\mathrm{PV}\begin{pmatrix}\text{one period before}\\ \text{first payment}\end{pmatrix} = \frac{1}{i-r}\left(1-\left(\frac{1+r}{1+i}\right)^{n}\right)$$

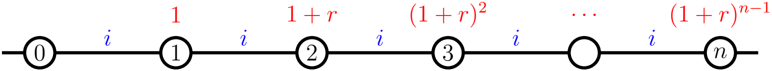


Pricing geometric annuities

Future value

Growth rate r, interest rate i, n periods:

$$\mathrm{FV}\begin{pmatrix}\text{time of the}\\ \text{last payment}\end{pmatrix} = \frac{1}{i-r}\left((1+i)^n - (1+r)^n\right)$$



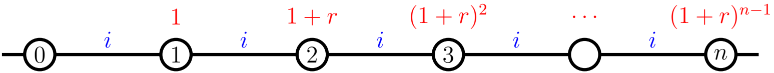
Lecture 21 March 6, 2023

Annuities with varying payments

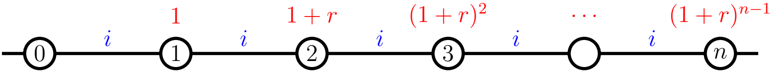
Present value

Growth rate r, interest rate i, n periods:

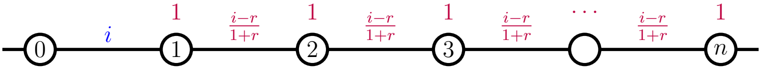
$$\text{PV}\begin{pmatrix}\text{one period before}\\ \text{first payment}\end{pmatrix} = \frac{1}{i-r}\left(1-\left(\frac{1+r}{1+i}\right)^n\right)$$



Pricing geometric annuities: using inflation



Payments that grow at a rate of r per period is equivalent to equal payments but at an inflation rate of r, the inflation-adjusted interest being $\dfrac{i-r}{1+r}$.



Example

You win \$10m at the Powerball. You either cash it in one lump sum or as a 30-year geometric annuity due with growth 5% (that pays a total of \$10m).

1. What are the amounts of the first and last payments? $(150514.35$ and $619537.48)$

Example

You win \$10m at the Powerball. You either cash it in one lump sum or as a 30-year geometric annuity due with growth 5% (that pays a total of \$10m).

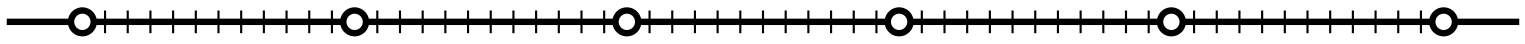
1. What are the amounts of the first and last payments? (150514.35 and 619537.48)
2. The tax rate is 55% for payments above 1m, and 20% for payments below 1m. The interest rate is 3%. Which option has the larger present value? (4.5m vs 4.7m)

Annual growth rate with monthly payments

2.3.1

Stan elects to receive his retirement benefit over 20 years at the rate of 2000 per month beginning one month from now. The monthly benefit increases by 5% each year. At a nominal interest rate of 6% convertible monthly, calculate the present value of the retirement benefit.

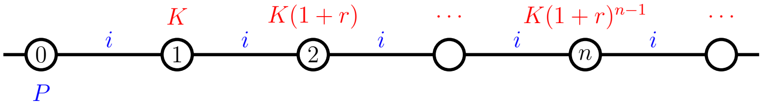
(24671 the first year, $i = 6.1677\%$, $\mathrm{PV} = 419242$)



Dividend Discount Model for Stock Prices

DDM

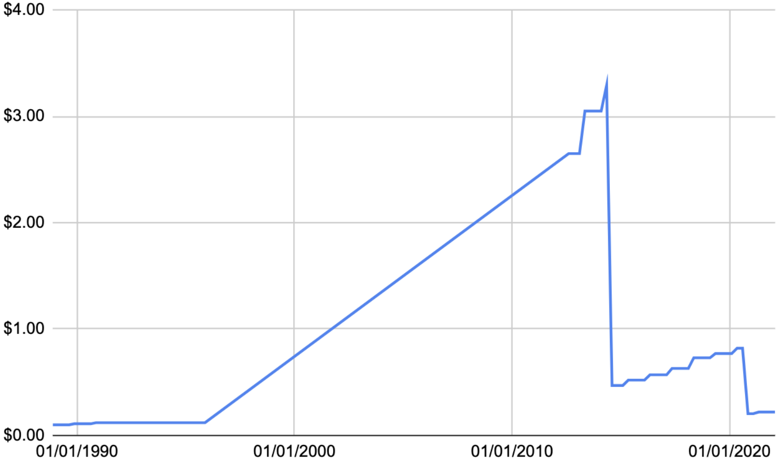
The Dividend Discount Model for pricing a stock posits that the price of the stock equals the present value of all future dividend payments.



If $r < i$:

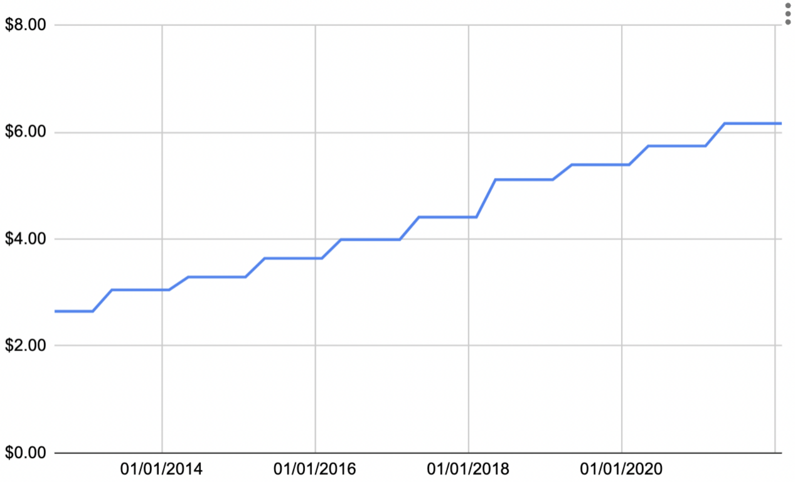
$$P = \lim_{n\to\infty} K \frac{1 - \left(\frac{1+r}{1+i}\right)^n}{i - r} = \frac{K}{i-r}.$$

Apple Stock Prices



Apple Stock Prices

Split-adjusted dividends for Apple:



Apple Stock Prices

Fiscal Year	May	Aug	Nov	Feb
2014-15	0.47	0.47	0.47	0.47
2015-16	0.52	0.52	0.52	0.52
2016-17	0.57	0.57	0.57	0.57
2017-18	0.63	0.63	0.63	0.63
2018-19	0.73	0.73	0.73	0.73
2019-20	0.77	0.77	0.77	0.77
2020-21	0.82	0.82	0.82	0.82
2021-22	0.88	0.88	0.88	0.88
2022-23	0.92	0.92	0.92	0.92

Apple Stock Prices

Fiscal Year	May	Aug	Nov	Feb
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2015-16	0.52	0.52	0.52	0.52
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2017-18	0.63	0.63	0.63	0.63
2018-19	0.73	0.73	0.73	0.73
2019-20	0.77	0.77	0.77	0.77
2020-21	0.82	0.82	0.82	0.82
2021-22	0.88	0.88	0.88	0.88
2022-23	0.92	0.92	0.92	0.92

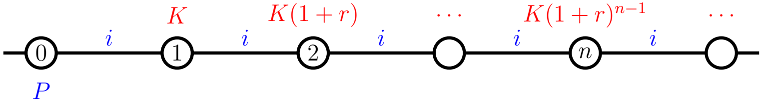
$$(1+r)^8 = \frac{0.92}{0.47} \qquad r = 8.76\%$$

$$K_q = \text{next quarterly dividend} = 0.92(1+r) = 1.$$

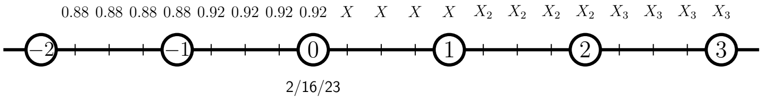
Lecture 22 March 8, 2023

Dividend Discount Model

$$P = \frac{K}{i - r}$$



Apple Dividends



Apple Stock Prices

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$$(1+r)^8 = \frac{0.92}{0.47} \qquad r = 8.76\%$$

$$K_q = \text{next quarterly dividend} = 0.92(1+r) = 1.$$

Apple Stock Prices on Feb 16, 2023

$$P = \frac{K}{i - r}$$

What is K here? It should be the next yearly dividend so

$$K = 4K_q s^{(4)}_{\overline{1}|i}$$

But what about the interest rate i?

Apple Stock Prices on Feb 16, 2023

$$P = \frac{K}{i - r}$$

What is K here? It should be the next yearly dividend so

$$K = 4K_q s^{(4)}_{\overline{1}|i}$$

But what about the interest rate i? Say we use $i = 10\%$. Then

$$K = 4K_q s^{(4)}_{\overline{1}|i} = 4.147$$

and so

$$\text{Apple Stock Price} = \frac{K}{i - r} = \frac{4.147}{10\% - 8.76\%} = 334.43.$$

At the time of the last dividend payment, Apple sold for 153.71. Clearly something is wrong.

Apple Stock Prices on Feb 16, 2023

$$P = \frac{K}{i - r}$$

What is K here? It should be the next yearly dividend so

$$K = 4K_q s^{(4)}_{\overline{1}|i}$$

Maybe we used the wrong i.

$$i = 4.75\%(\text{e.g., US Discount Rate in } 3/2023)$$

K is the next yearly dividend, in our model:

$$K = 4K_q s^{(4)}_{\overline{1}|i} = 4.07056.$$

So we get the price

$$\text{Apple Stock Price} = \frac{K}{i - r} = \frac{4.07056}{4.75\% - 8.76\%} = -101.51.$$

Apple Stock Prices on Feb 16, 2023

$$P = \frac{K}{i - r}$$

Implied Interest Rate

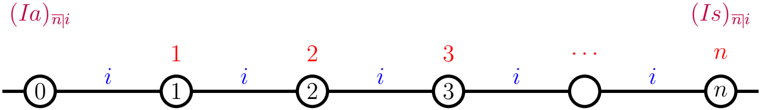
Instead, we'll compute a value for the interest rate i which makes the stock price equal to the DDM price. This is the **implied interest rate** from DDM.

Apple stock price is 153.71 on 2/16/2023:

$$\frac{K}{i - r} = 153.71 \qquad \frac{4 s^{(4)}_{\overline{1}|i}}{i - r} = 153.71.$$

$$i = \text{Implied Interest Rate} = 9.43\%.$$

Pricing arithmetic annuities



Increasing Arithmetic Annuities

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i}$$

Portfolio:

- Buy an $\mathrm{IAnnuity}_{\overline{n}|i}$ at time 0,
- Sell an $\mathrm{IAnnuity}_{\overline{n}|i}$ at time 1.

Price the portfolio in two different ways to find $Ia_{\overline{n}|i}$.

Exercise

2.3.14

A perpetuity costs 77.1 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year 3,…,n at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n. The annual effective interest rate is 10.5%. Calculate n.

$(n = 19)$

Exercise

2.3.14

A perpetuity costs 77.1 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year 3,…,n at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n. The annual effective interest rate is 10.5%. Calculate n.

$(n = 19)$

$$77.1 = \nu (Ia)_{\overline{n-1}|i} + n a_{\overline{\infty}|i} \nu^n$$

$$= \nu \frac{\ddot{a}_{\overline{n-1}|i} - (n-1)\nu^{n-1}}{i} + \frac{n\nu^n}{i}$$

$$= \frac{1}{i}\left(\nu \frac{1-\nu^{n-1}}{1-\nu} - (n-1)\nu^n + n\nu^n\right)$$

$$77.1 = \frac{\nu}{i(1-\nu)}(1-\nu^n)$$

Lecture 23 March 10, 2023

US Government budget and GDP

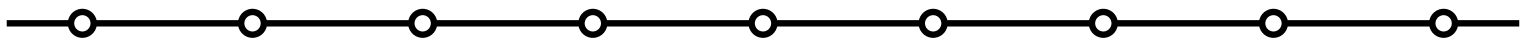
(fiscaldata.treasury.gov)

Exercise: takes 2 and 3

2.3.14

A perpetuity costs 77.1 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year 3,…,n at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n. The annual effective interest rate is 10.5%. Calculate n.

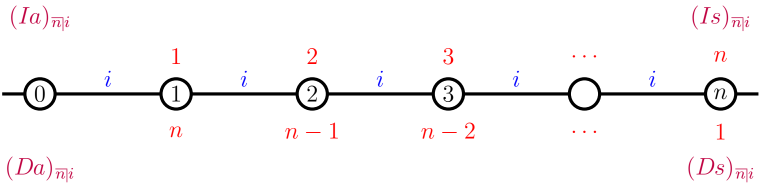
$(n = 19)$



$$77.1 = \nu(Ia)_{\overline{n}|i} + n a_{\overline{\infty}|i} \nu^{n+1}$$

$$77.1 = n a_{\overline{\infty}|i} - (Da)_{\overline{n}|i}$$

Pricing Decreasing Arithmetic Annuities



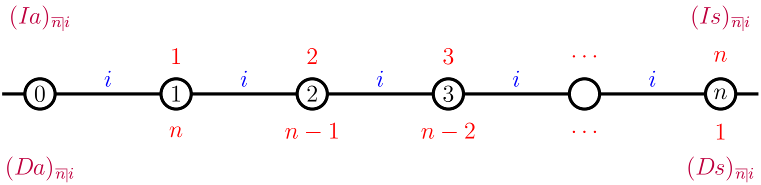
$$I\,\text{Annuity}_{\overline{n}|i} + D\,\text{Annuity}_{\overline{n}|i} = (n+1)\,\text{Annuity}_{\overline{n}|i}$$

$$(Ia)_{\overline{n}|i} + (Da)_{\overline{n}|i} = (n+1)a_{\overline{n}|i}$$

Recalling that

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i}$$

Pricing Decreasing Arithmetic Annuities



$$I\,\text{Annuity}_{\overline{n}|i} + D\,\text{Annuity}_{\overline{n}|i} = (n+1)\,\text{Annuity}_{\overline{n}|i}$$

$$(Ia)_{\overline{n}|i} + (Da)_{\overline{n}|i} = (n+1)a_{\overline{n}|i}$$

Recalling that

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - n\nu^n}{i}$$

$$(Ia)_{\overline{\infty}|i} = \frac{\ddot{a}_{\overline{\infty}|}}{i} = \frac{1}{i} + \frac{1}{i^2}.$$

Continuous annuities with nonlevel payments

We already discussed continuous annuities with nonlevel interest rates (i.e., with nonlevel force of interest).

Continuous Annuities

Suppose an annuity pays $h(t)$ at time t between $0 \leq t \leq n$. Then

$$\mathrm{PV}_{t=0} = \int_0^n h(t) \frac{1}{a(t)} dt,$$

where $a(t) = e^{\delta t}$ if δ is constant.

1. Level annuity has $h(t) = K$ constant.
2. Geometric annuity has $h(t) = K(1+r)^t$.
3. Increasing (resp. descreasing) arithmetic annuity has $h(t) = Kt$ (resp. $h(t) = K(n-t)$).

Pricing continuous annuities

Continuous annuity price

$$\mathrm{PV}_{t=0} = \int_0^n \frac{h(t)}{a(t)} dt.$$

Geometric annuities $h(t) = (1+r)^t$, $a(t) = e^{\delta t}$

$$\mathrm{PV}_{t=0} = \int_0^n (1+r)^t e^{-\delta t} dt = \int_0^n (e^{\ln(1+r)})^t e^{-\delta t} dt$$

Pricing continuous annuities

Continuous annuity price

$$\mathrm{PV}_{t=0} = \int_0^n \frac{h(t)}{a(t)} dt.$$

Geometric annuities $h(t) = (1+r)^t$, $a(t) = e^{\delta t}$

$$\begin{aligned}\mathrm{PV}_{t=0} &= \int_0^n (1+r)^t e^{-\delta t} dt = \int_0^n (e^{\ln(1+r)})^t e^{-\delta t} dt \\ &= \int_0^n e^{t(\ln(1+r)-\delta)} dt = \frac{e^{n(\ln(1+r)-\delta)} - 1}{\ln(1+r) - \delta}\end{aligned}$$

Pricing continuous annuities

Continuous annuity price

$$\mathrm{PV}_{t=0} = \int_0^n \frac{h(t)}{a(t)} dt.$$

Geometric annuities $h(t) = (1+r)^t$, $a(t) = e^{\delta t}$

$$\begin{aligned}
\mathrm{PV}_{t=0} &= \int_0^n (1+r)^t e^{-\delta t} dt = \int_0^n (e^{\ln(1+r)})^t e^{-\delta t} dt \\
&= \int_0^n e^{t(\ln(1+r)-\delta)} dt = \frac{e^{n(\ln(1+r)-\delta)} - 1}{\ln(1+r) - \delta} \\
&= \frac{e^{n \ln \frac{1+r}{1+i}} - 1}{\ln \frac{1+r}{1+i}} = \frac{\left(\frac{1+r}{1+i}\right)^n - 1}{\ln \frac{1+r}{1+i}}
\end{aligned}$$

Pricing continuous annuities

Continuous annuity price

$$\text{PV}_{t=0} = \int_0^n \frac{h(t)}{a(t)} dt.$$

Increasing arithmetic annuities $h(t) = t$, $a(t) = e^{\delta t}$

$$\begin{aligned}
\text{PV}_{t=0} &= \int_0^n t e^{-\delta t} dt = \int_0^n t d\left(\frac{e^{-\delta t}}{-\delta}\right) \\
&= t\left(\frac{e^{-\delta t}}{-\delta}\right)\Big|_0^n - \int_0^n \left(\frac{e^{-\delta t}}{-\delta}\right) dt \\
&= t\left(\frac{e^{-\delta t}}{-\delta}\right)\Big|_0^n - \left(\frac{e^{-\delta t}}{\delta^2}\right)\Big|_0^n \\
&= \frac{1}{\delta^2} - \frac{n e^{-\delta n}}{\delta} - \frac{e^{-\delta n}}{\delta^2}
\end{aligned}$$

Exercise

A financial instrument consists of an initial payment of 100 and then a continuous annuity that pays at the level $h(t) = 20 + 5t + 1.03^t$, with force of interest $\delta = 5\%$. How much is accumulated in the account after 10 years?