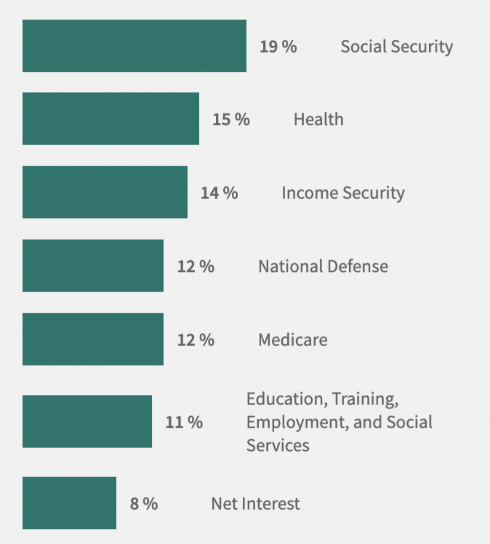
Lecture 23 March 20, 2023

US Government budget

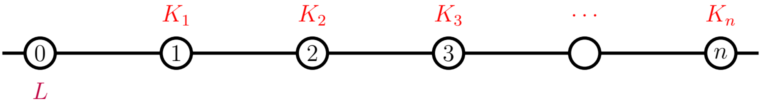


(fiscaldata.treasury.gov)

Loans and Amortizations

Amortization

An **amortization** for a loan is a repayment of the loan over several installments.



Questions:

- How to determine the value of the loan given the payments?
- How to structure the payments to repay the loan exactly?
- How much money is still owed after a certain time?

Loans and Amortizations

Conceptual Answers

- To determine the relation between L and $K_1, K_2, \ldots, K_n$ we write the **equation of value**.
- For determining how much is still owed, the rule of thumb is that **interest gets paid first**.

Loans and Amortizations

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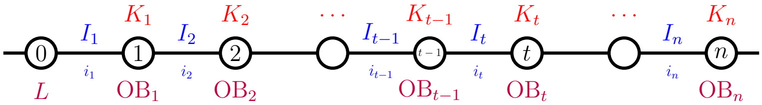
Example

1. (Lump sum loan) $K_1 = K_2 = \ldots = K_{n-1} = 0$ and $K_n > 0$.
2. (Level payment loan) $K_1 = K_2 = \ldots = K_n$ (mortgages, car loans).
3. (Variable interest rates) The interest rate can change from payment to payment.

Outstanding Balance, Interest, Principal Repaid

Terminology



- Interest rate i_t from time $t-1$ to time t.
- Outstanding balance OB_t at time t, i.e., how much is still owed on the loan **after** K_t. $\mathrm{OB}_0 = L$ and $\mathrm{OB}_n = 0$.

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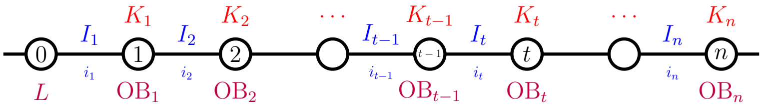

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- Interest I_t, accrued from time $t-1$ to time t.

$$I_t = \mathrm{OB}_{t-1} \cdot i_t.$$

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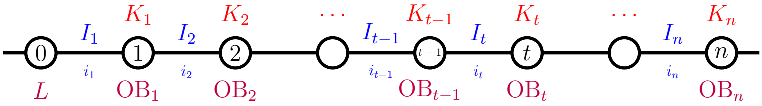

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- Interest added to balance, then payment deducted:

$$\mathrm{OB}_t = \mathrm{OB}_{t-1} + I_t - K_t.$$

Outstanding Balance, Interest, Principal Repaid

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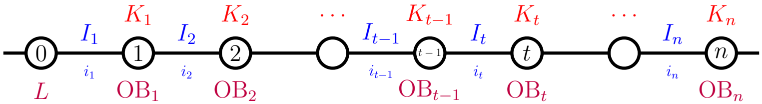

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- Interest added to balance, then payment deducted:

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- Principal repaid is the portion of the payment that's not to cover interest ("interest paid first"):

$$\mathrm{PR}_t = K_t - I_t.$$

Equations of Value

There are two ways to set up the equation of value:

- **Forward looking EoV**: the OB equals the present value of all future payments.

$$\begin{aligned}\mathrm{OB}_t &= \mathrm{PV}_t(K_{t+1}) + \cdots + \mathrm{PV}_t(K_n)\\ &= K_{t+1}\nu + \cdots + K_n\nu^{n-t}.\end{aligned}$$

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- **Backward looking EoV**: the OB equals the present value of the initial loan minus the present value of all previous payments.

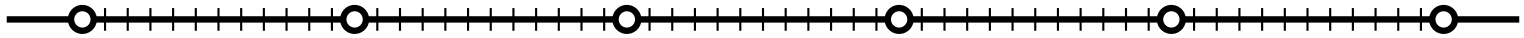
$$\begin{aligned}\mathrm{OB}_t &= \mathrm{FV}_t(L) - \mathrm{FV}_t(K_1) - \cdots - \mathrm{FV}_t(K_t) \\ &= L(1+i)^t - K_1(1+i)^{t-1} - \cdots - K_{t-1}(1+i) - K_t.\end{aligned}$$

Example

3.1.2

A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is 1000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment. Calculate the outstanding loan balance immediately after the 40^{th} payment is made.

$(\mathrm{OB}_{40} = 6889.11$ and $L = 29452.83)$



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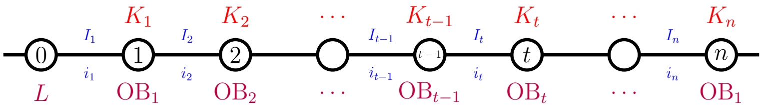
Amortizations



$$\begin{aligned} I_t &= \mathrm{OB}_{t-1} \cdot i_t \\ \mathrm{PR}_t &= K_t - I_t \\ \mathrm{OB}_t &= \mathrm{OB}_{t-1} - \mathrm{PR}_t = \mathrm{OB}_{t-1} + I_t - K_t \end{aligned}$$

- (Forward) $\mathrm{OB}_t = K_{t+1}\nu + K_{t+2}\nu^2 + \cdots + K_n\nu^{n-t}$.
- (Backward) $\mathrm{OB}_t = L(1+i)^t - K_1(1+i)^{t-1} - \cdots - K_t$.

Exercise

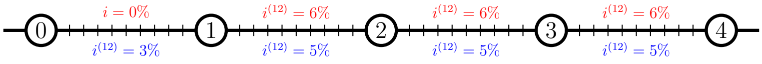
3.1.4

Smith borrows 20,000 to purchase a car. The car dealer finances the purchase and offers Smith two alternative financing plans, both of which require monthly payments at the end of each month for 4 years starting one month after the car is purchased.

(i) 0% interest rate for the first year followed by 6% nominal annual interest rate compounded monthly for the following three years.

(ii) 3% nominal annual interest rate compounded monthly for the first year followed by 5% nominal annual interest compounded monthly for the following three years.

For each of (i) and (ii) find the monthly payment and the outstanding balance on the loan at the end of the first year.



Exercise

Say X and Y are the level payments for the two options. The forward equations are

$$\begin{aligned}
20k &= \mathrm{PV}_{t=0}(\mathrm{Annuity}_{\overline{12}|0\%}) + \mathrm{PV}_{t=0}(\mathrm{Annuity}_{\overline{36}|\frac{6\%}{12}}) \\
&= X a_{\overline{12}|0\%} + \nu_{0\%}^{12} X a_{\overline{36}|0.5\%} \\
&= X \left(a_{\overline{12}|0\%} + \nu_{0\%}^{12} a_{\overline{36}|0.5\%} \right) \\
X &= 445.72 \\
20k &= \mathrm{PV}_{t=0}(\mathrm{Annuity}_{\overline{12}|\frac{3\%}{12}}) + \mathrm{PV}_{t=0}(\mathrm{Annuity}_{\overline{36}|\frac{5\%}{12}}) \\
&= Y a_{\overline{12}|0.25\%} + \nu_{0.25\%}^{12} Y a_{\overline{36}|\frac{5\%}{12}} \\
&= Y \left(a_{\overline{12}|0.25\%} + \nu_{0.25\%}^{12} a_{\overline{36}|\frac{5\%}{12}} \right) \\
Y &= 452.61
\end{aligned}$$

Amortization Table

An **amortization table** is a table containing OB, I, K, PR at each time period.

t	i_t	I_t	K_t	PR_t	OB_t
0	$-$	$-$	$-$	$-$	L
1	i_1	I_1	K_1	$K_1 - I_1$	$L - \mathrm{PR}_1$
			$\vdots$		
t	i_t	$\mathrm{OB}_{t-1} \cdot i_t$	K_t	$\mathrm{PR}_t = K_t - I_t$	$\mathrm{OB}_{t-1} - \mathrm{PR}_t$
			$\vdots$		
n	i_n	$\mathrm{OB}_{n-1} \cdot i_n$	K_n	$\mathrm{PR}_n = K_n - I_n$	0

T stands for **total**:

$$K_T = K_1 + K_2 + \cdots + K_n \qquad I_T = I_1 + I_2 + \cdots + I_n$$

$$\mathrm{PR}_T = \mathrm{PR}_1 + \mathrm{PR}_2 + \cdots + \mathrm{PR}_n \qquad \mathrm{PR}_T = K_T - I_T = L$$

Basic example

A 3 year loan has interest rates 5%, 15%, and 10%, and yearly payments of 100, 300, 200. What is the amount of the loan? Construct the amortization table.

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$$
\begin{aligned}
L &= \frac{K_1}{a(1)} + \frac{K_2}{a(2)} + \frac{K_3}{a(3)} \\
&= \frac{100}{1.05} + \frac{300}{1.05 \cdot 1.15} + \frac{200}{1.05 \cdot 1.15 \cdot 1.1} \\
&= 494.26
\end{aligned}
$$

t	i_t	I_t	K_t	PR_t	OB_t
0					494.26
1	0.05	24.71	100	75.29	418.97
2	0.15	62.85	300	237.15	181.82
3	0.1	18.18	200	181.82	0

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t	i_t	I_t	K_t	PR_t	OB_t
0	$-$	$-$	$-$	$-$	L
1	i_1	I_1	K_1	$K_1 - I_1$	$L - \mathrm{PR}_1$
			$\vdots$		
t	i_t	$\mathrm{OB}_{t-1} \cdot i_t$	K_t	$\mathrm{PR}_t = K_t - I_t$	$\mathrm{OB}_{t-1} - \mathrm{PR}_t$
			$\vdots$		
n	i_n	$\mathrm{OB}_{n-1} \cdot i_n$	K_n	$\mathrm{PR}_n = K_n - I_n$	0

Example

A 100k loan is being repaid in 10 years, with level annual payments beginning one year after the time of the loan. The load carries an adjustable interest rate, with 3% for the first 5 years, and thereafter indexed to a standard interest rate index R. At the time of the loan, the index R is projected to be $(2 + t/5)\%$ over the t-th year. What is the amortization table?

t	i_t	$a(t)$
1	3.0%	1.03000000000000
2	3.0%	1.06090000000000
3	3.0%	1.09272700000000
4	3.0%	1.12550881000000
5	3.0%	1.15927407430000
6	3.2%	1.19637084467760
7	3.4%	1.23704745339664
8	3.6%	1.28158116171892
9	3.8%	1.33028124586424
10	4.0%	1.38349249569881

Example

t	i_t	I_t	K_t	PR_t	OB_t
0	%				100000.0
1	3.0%	3000.0	11794.18	8794.18	91205.82
2	3.0%	2736.17	11794.18	9058.0	82147.82
3	3.0%	2464.43	11794.18	9329.74	72818.08
4	3.0%	2184.54	11794.18	9609.63	63208.44
5	3.0%	1896.25	11794.18	9897.92	53310.52
6	3.2%	1705.94	11794.18	10088.24	43222.28
7	3.4%	1469.56	11794.18	10324.62	32897.66
8	3.6%	1184.32	11794.18	10609.86	22287.8
9	3.8%	846.94	11794.18	10947.24	11340.56
10	4.0%	453.62	11794.18	11340.56	0

Example

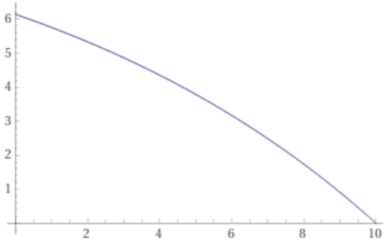
Imagine now that you are at the end of the 5th year, and interest rates went through the roof, they swing wildly between 3% and 7%. You've been paying 11794.18 yearly, because that's what the expected rate was to repay the loan exactly. At year 5, you have the option to switch from R to constant 5% interest rate for the remainder of the loan. What would the level payments be with this new fixed interest rate? (12313.39)

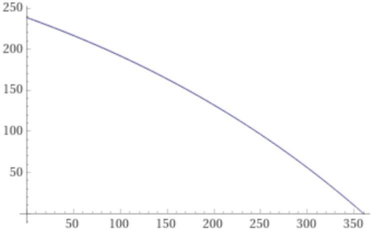
Amortizations with level payments

If a loan of L is repaid with level payments at regular intervals and constant interest rate i then.

$$L = K\,a_{\overline{n}|i} = K\frac{1-\nu^n}{i}$$

$$\mathrm{OB}_t = K\,a_{\overline{n-t}|i} = K\frac{1-\nu^{n-t}}{i}$$





Amortization table with level payments

Level payment K, level interest rate i.

t	i_t	I_t	K_t	PR_t	OB_t
0					$K a_{\overline{n}\vert i}$
1	i	$K(1-\nu^n)$	K	$K\nu^n$	$K a_{\overline{n-1}\vert i}$
			$\vdots$		
t	i	$K(1-\nu^{n-t+1})$	K	$K\nu^{n-t+1}$	$K a_{\overline{n-t}\vert i}$
			$\vdots$		
n	i	$K(1-\nu)$	K	$K\nu$	0

Exercise

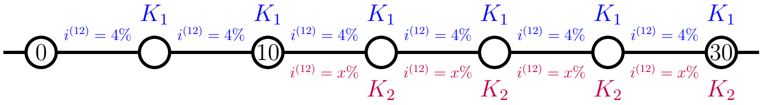
3.2.17

A loan is being repaid by $2n$ level payments, starting one year after the loan. Just after the n^{th} payment the borrower finds that she still owes $\frac{3}{4}$ of the original amount. What proportion of the next payment is interest?

$(2/3)$

Mortgage refinancing

A 30-year 250k mortgage is taken out at 4% nominal monthly. After 10 years, an opportunity arises to refinance the balance on the original mortgage as a 20-year mortgage. For what nominal interest rate does it make financial sense to refinance, knowing there is a 4k refinancing charge?



$(K_1 = 1193.54,\ L_2 = 200959.9,\ x < 3.713\%)$.