The Term Structure of Interest Rates

What affects interest rates?

- Risk for a specific investment.
- Collaterals.
- Prevailing cost of capital.
- Length of time.

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Definition

A **term structure** is a relationship between the term of a loan and its interest rate.

The US Treasury Yield Curve

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
03/01/2022	0.11	0.21	0.32	0.60	0.91	1.31	1.47	1.56	1.67	1.72	2.19	2.11
03/02/2022	0.13	0.24	0.34	0.68	1.06	1.50	1.67	1.74	1.83	1.86	2.32	2.24
03/03/2022	0.19	0.25	0.38	0.69	1.08	1.53	1.69	1.74	1.82	1.86	2.32	2.24
03/04/2022	0.15	0.21	0.34	0.69	1.05	1.50	1.62	1.65	1.70	1.74	2.23	2.16
03/07/2022	0.17	0.23	0.38	0.75	1.07	1.55	1.68	1.71	1.77	1.78	2.29	2.19

Date	1 Mo	2 Mo	3 Mo	4 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
04/03/2023	4.70	4.79	4.90	4.98	4.88	4.60	3.97	3.73	3.52	3.48	3.43	3.78	3.64
04/04/2023	4.66	4.80	4.88	4.90	4.80	4.50	3.84	3.60	3.39	3.38	3.35	3.72	3.60
04/05/2023	4.62	4.77	4.86	4.90	4.82	4.43	3.79	3.55	3.36	3.34	3.30	3.67	3.56
04/06/2023	4.57	4.85	4.91	4.98	4.93	4.51	3.82	3.59	3.37	3.34	3.30	3.66	3.54
04/07/2023	4.56	4.90	4.95	5.07	4.95	4.61	3.97	3.72	3.49	3.45	3.39	3.73	3.61

The US Treasury Yield Curve

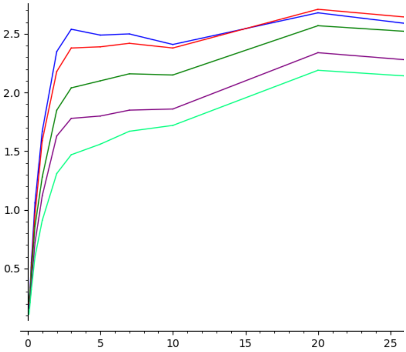


Figure: YC 3/22

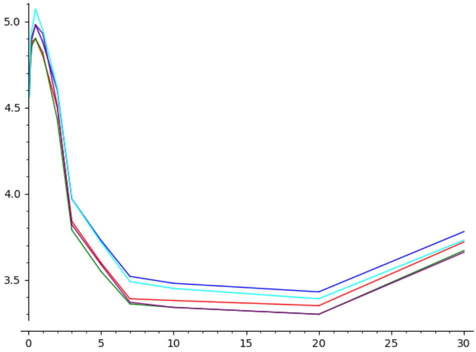


Figure: YC 4/23

Yield Curves vs Credit Rating

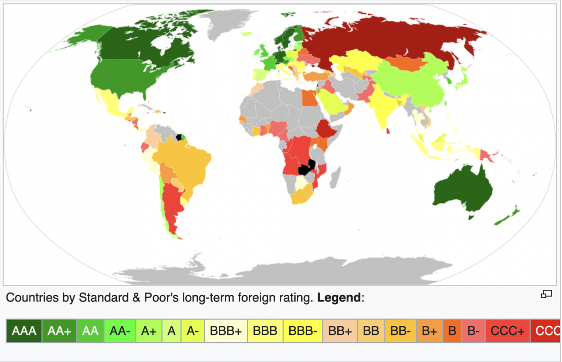
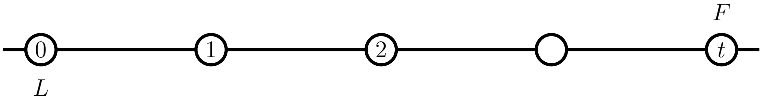


Figure: S&P Credit Rating, Wikipedia 3/22

Term Structure as a Function

Definition

We denote $s_0(t)$ the effective annual interest rate for borrowing money at time 0 to be repaid at time t.



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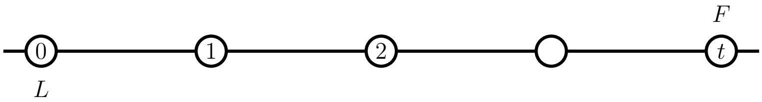
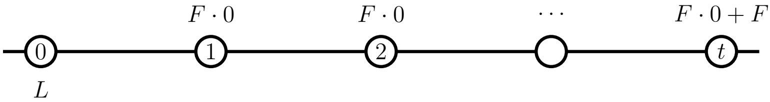


Remark

A loan to be repaid in full at term is a 0-coupon bond.

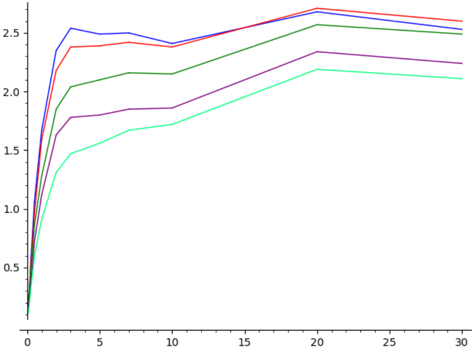


$s_0(t)$ is the yield rate of a 0-coupon bond with term t.

Term structure of spot rate and the Yield Curve

The term structure of interest rates is the collection $s_0(t)$ of 0-coupon yield rates. The yield rates $s_0(t)$ are called **spot rates of interest**.

The term structure depends on when the present, i.e., $t = 0$, occurs, and it typically changes from day to day.



Pricing Assets with the Yield Curve

If interest rates are smaller for short term loans than for long term loans, we should discount future money more if farther in the future.

- 0-coupon bond with term t
- Accumulation factor at time t
- Net Present Value

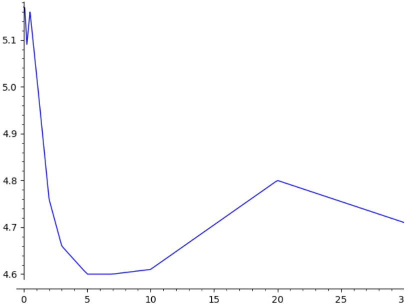


Figure: YC 11/13/06

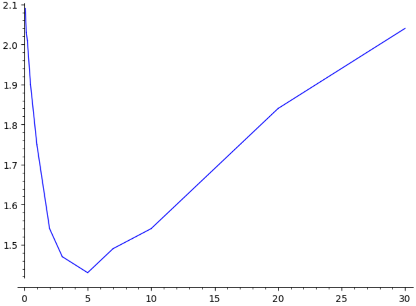
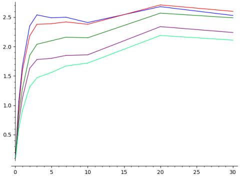


Figure: YC 8/26/19

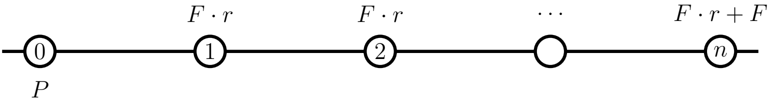
Usual YC (March 2022)



Pricing Bonds with the Yield Curve

Example

Suppose we have a standard bond with F, r, n. If the term structure is $(s_0(t))$, how should we price this bond?



Exercise

6.1.1

You are given the following term structure:

$$r_1 = .15, \qquad r_2 = .10, \qquad r_3 = .05.$$

These are *effective annual rates of interest* for zero coupon bonds of 1, 2 and 3 years maturity, respectively. A newly issued 3-year bond with face amount 100 has annual coupon rate 10%, with coupons paid *once per year* starting one year from now.

Find the price and effective annual yield to maturity of the bond.

(111.98, 5.56%)

Lecture 35 April 19, 2023

Term Structure

Interest rate depends on the **term** with $s_0(t)$ the annual interest rate for a loan taken now and due at time t.

$$a(t) = (1 + \underbrace{i}_{s_0(t)})^t$$

The term structure enters computations ONLY via $a(t)$.

Bond example

For Immediate Release
March 27, 2023

CONTACT: Treasury Auctions
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	2-Year Note
CUSIP Number	91282CGU9
Series	AZ-2025
Interest Rate	3-7/8%
High Yield [1]	3.954%
Allotted at High	13.09%
Price	99.849511
Accrued Interest per $1,000	None
Median Yield [2]	3.870%
Low Yield [3]	3.800%
Issue Date	March 31, 2023

Type	Yield	Price
High yield/Low price	3.954%	99.849511
Median yield/Median price	3.870%	100.009534
Low yield/High price	3.800%	100.143137

Bond example

Yield curve on 3/27/2023

1m	2m	3m	4m	6m	1y	2y
4.22	4.47	4.91	4.90	4.86	4.51	3.94

3y	5y	7y	10y	20y	30y
3.79	3.59	3.57	3.53	3.90	3.77

Computed price of 3/27/2023 bond:

$$P = 99.922638$$

Constructing the Yield Curve

If we know the term structure $(s_0(t))$ for $t \leq n$ we can price annual bonds with term n:

$$P_{F,r,n} = \frac{Fr}{1 + s_0(1)} + \frac{Fr}{(1 + s_0(2))^2} + \cdots + \frac{Fr + F}{(1 + s_0(n))^n}.$$

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Yield Curve from Bonds

If we looked at the prices of many bonds of different kinds of terms, we could obtain many price equations and **deduce** the term structure of interest rates, in other words we can construct the yield curve.

Exercise

6.1.5

You are given the following information for 4 bonds. All coupon and yield-to-maturity rates are nominal annual convertible twice per year.

Bond	Time to Maturity	Coupon Rate	YTM
1	½-year	4%	.05
2	1-year	6%	.10
3	1½-year	4%	.15
4	2-year	8%	.15

Find the associated term structure for zero coupon bonds with maturities of ½-year, 1-year, 1½-year, and 2-year (quotations should be nominal annual rates convertible twice per year).

(

Exercise

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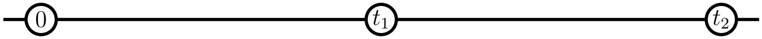
4	2-year	8%	.15

Find the associated term structure for zero coupon bonds with maturities of ½-year, 1-year, 1½-year, and 2-year (quotations should be nominal annual rates convertible twice per year).

(Prices 99.512195, 96.281179, 85.697108, 88.277358, Spot rates 5%, 10.078%, 15.151%, 15.23%)

Forward Interest Rates

Suppose we are given a yield curve/term structure $(s_0(t))$. The **forward interest rate** between t_1 and t_2 is the implied constant interest rate $i(t_1, t_2)$ for a loan between t_1 and t_2.



Implied by what? By

$$a(t_1)a(t_1, t_2) = a(t_2)$$

Forward Interest Rates

Suppose we are given a yield curve/term structure $(s_0(t))$. The **forward interest rate** between t_1 and t_2 is the implied constant interest rate $i(t_1, t_2)$ for a loan between t_1 and t_2.

Implied by what? By

$$a(t_1)a(t_1, t_2) = a(t_2)$$

$$(1 + s_0(t_1))^{t_1}(1 + i(t_1, t_2))^{t_2 - t_1} = (1 + s_0(t_2))^{t_2}$$

Exercise

6.3.5

The following term structure is given as effective annual rates of interest on zero coupon bonds:

1-year maturity: 6% 2-year maturity: 7% 3-year maturity: 9%

(a) Find (i) the 1-year forward effective annual interest rate for a 1-year period, $f_{[1,2]}$ and (ii) the 2-year forward effective annual interest rate for a 1-year period, $f_{[2,3]}$.

(b) The effective annual rate of interest for a 4-year zero coupon bond is r_4. Find the minimum value of r_4 needed so that $f_{[3,4]} \geq f_{[2,3]}$, where $f_{[3,4]}$ is the 3-year forward effective annual interest rate for a 1-year period and $f_{[2,3]}$. is found in part (a).

(8.01%, 13.11%, and $\geq 10.01\%$)

Lecture 36 April 13, 2022

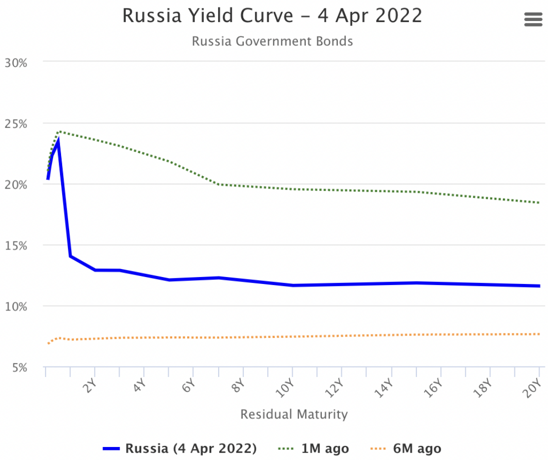


Figure: $D_{0.5}(\text{Russia vs US}) = 9.5\%$

Exercise

The term structure of interest rates in the capital markets is currently the following

0.5	5%
1	7%
1.5	9%
2	11%
2.5	9%
≥ 3	7%

What is the 1-year forward yield curve?

1-year forward 0.5	
1-year forward 1	
1-year forward 1.5	
1-year forward 2	
1-year forward 2.5	
1-year forward ≥ 3	

Exercise

The term structure of interest rates in the capital markets is currently the following

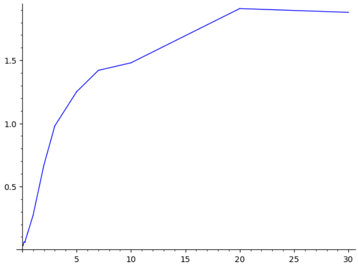
0.5	5%
1	7%
1.5	9%
2	11%
2.5	9%
≥ 3	7%

What is the 1-year forward yield curve?

1-year forward 0.5	13.11%
1-year forward 1	15.15%
1-year forward 1.5	10.35%
1-year forward 2	7%
1-year forward 2.5	7%
1-year forward ≥ 3	7%

Yield Curve as a function from 0 to ∞

Suppose you are quoted a term structure/yield curve $(s_0(t))$ at a discrete set of points? How do you interpolate in between these dates?



A convenient way is to interpolate linearly.

Interest Rate model from Yield Curve

Setup

We would like to model the future behavior of interest rates over short periods of time. Say i_t is the (effective annual) interest rate that we will pay at time t for an overnight loan.

Model:

$$i_t = i_{\text{forward}}(t, t + \underbrace{1 \text{ day}}_{h}).$$

Solve:

$$(1 + s_0(t+h))^{t+h} = (1 + s_0(t))^t (1 + i_t)^h$$

Interest Rate model from Yield Curve: 3/29/2022

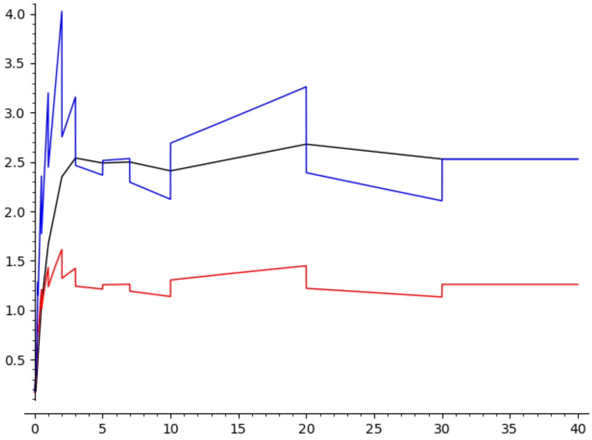


Figure: Interest Rate Model from Yield Curve

Interest Rate Spreads

A can borrow at i_A, B can borrow at i_B.
The spread $i_A - i_B$ measures the excess risk for lending to A compared to B.

- Governments borrowing from the markets (bonds)
- Banks borrowing from governments
- Banks borrowing from banks
- Investors borrowing from banks
- Companies borrowing from the markets (bonds)
- Persons borrowing from banks

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- Banks borrowing from governments
- Banks borrowing from banks
- Investors borrowing from banks
- Companies borrowing from the markets (bonds)
- Persons borrowing from banks
- Implied interest rates without any borrowing happening

Example

We computed that on $2/16/2023$ Apple stocks had an implied IR from DDM of 9.43% compared to 3.92% long term for US Bonds.

Interest Rate Spreads

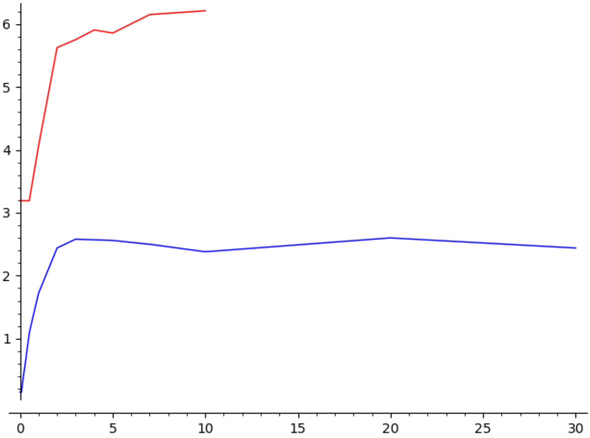


Figure: YC Spread US vs Romania 4/4/22

Dividend Discount Model: Take 2

The DDM prices a stock as the PV of all future dividend payments $P = \sum_{t=1}^{\infty} \mathrm{PV}(d_t)$.

In a world where investors are **risk neutral**, i.e., they don't care about the extra risk of investing in stock compared to the safety of the US Treasury yield curve $s_0(t)$:

$$P = \sum_{t=1}^{\infty} \frac{d_t}{(1 + s_0(t))^t}.$$

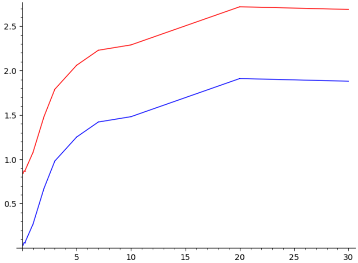
We've already seen that Apple is charged for excess risk!

Apple Stock Price on 2/16/2023

The discrepancy between the price of stock and the DDM price can be interpreted as saying that the market assesses an **excess risk penalty** for Apple vs US.

Let's compute this **excess risk** as the value of x such that term structure $s_W(t)$ that we use to compute PV of Walmart dividends is a **parallel shift** of the US Treasury yield curve:

$$s_W(t) = s_0(t) + x, \text{ for all } t.$$



Tesla Interest Rate Spread

Tesla hasn't paid dividends yet, so how could we possible study its stock price? Instead of DDM we'll use **bonds** to price Tesla's IR spread.

Bond Issuer	Tesla Inc	Bond Currency	USD	Issue Size	1,800,000,000	Bond Credit Rating (S&P/ Fitch)	BB/ N.R
Guarantor	Tesla Energy Operations Inc/DE	Annual Coupon Rate (% p.a.)	5.300	Announcement Date	11 Aug 2017	Issuer Credit Rating (S&P/ Fitch)	BB/ N.R
ISIN	USU8810LAA18	Coupon Type	Fixed	Issue Date	18 Aug 2017	Exchange Listed	Others
CUSIP	AO7577130	Reference Rate	-	Maturity Date	15 Aug 2025	Seniority	Senior Unsecured
Bond Type	High Yield Corporate	Annual Coupon Frequency	Semi Annually	Years to Maturity	4.326	Sukuk Investing	Conventional
Bond Sector	Consumer Discretionary	Min. Investment Quantity	USD 2,000	Issue / Reoffer Price	100.000		
Bond Sub Sector	Automobiles	Incremental Quantity	USD 1,000				

Figure: TSLA 1.8b bond maturing 8/15/25, called on 8/15/21

Pricing TSLA Bond on 4/16/21

TSLA 5.300% 15Aug2025 Corp (USD)

Tesla Inc

Latest Quotes as of 16 Apr 2021

Tesla Inc. designs, manufactures, and sells high-performance electric vehicles and electric vehicle powertrain components. The Company owns its sales and service network and sells electric powertrain components to other automobile manufacturers. Tesla serves customers worldwide.

Ask Yield to Maturity (% p.a)	Ask Yield to Worst (% p.a)	Indicative Ask Price USD	Bond Complexity
4.216	0.452	104.245	Moderate
Bid Yield to Maturity (% p.a)	**Bid Yield to Worst (% p.a)**	**Indicative Bid Price USD**	**Investor Profile**
4.299	1.417	103.911	High Yield Seeker

Price (bid) of the bond using US YC

$$P = \frac{Fr}{(1+s_0(t_1))^{t_1}} + \frac{Fr}{(1+s_0(t_2))^{t_2}} + \cdots + \frac{Fr+F}{(1+s_0(t_n))^{t_n}}$$

Price (bid) of the bond using parallel shift up of the US YC

$$P = \frac{Fr}{(1+s_0(t_1)+r)^{t_1}} + \frac{Fr}{(1+s_0(t_2)+r)^{t_2}} + \cdots + \frac{Fr+F}{(1+s_0(t_n)+r)^{t_n}}$$

Pricing TSLA Bond on 4/16/21

	A	B	C	D	E	F	G	H	I	J	K
1	**YC**	**04/16/21**			TSLA	5.30%	Bid price	103.911		**r**	3.88%
2	**t**	**t in years**	**YC rate**		**Coupons**	**Coupon Amt**	**Time until pay**	**YC rate at pay**	**PV @ YC**	**YC+r**	**PV @ YC+r**
3	1m	0.083333333	0.02%		8/15/2021	2.65	0.3315068493	0.03%	2.649767044	3.91%	2.61621789
4	2m	0.166666666	0.02%		2/15/2022	2.65	0.8356164384	0.05%	2.648817395	3.93%	2.565104633
5	3m	0.25	0.02%		8/15/2022	2.65	1.331506849	0.09%	2.646715986	3.97%	2.514710936
6	6m	0.5	0.04%		2/15/2023	2.65	1.835616438	0.14%	2.643028303	4.02%	2.463076788
7	1y	1	0.06%		8/15/2023	2.65	2.331506849	0.22%	2.636469735	4.10%	2.410685519
8	2y	2	0.16%		2/15/2024	2.65	2.835616438	0.31%	2.62679481	4.19%	2.355907132
9	3y	3	0.34%		8/15/2024	2.65	3.334246575	0.42%	2.612877185	4.30%	2.299165507
10	5y	5	0.84%		2/15/2025	2.65	3.838356164	0.55%	2.594758338	4.43%	2.239697985
11	7y	7	1.26%		8/15/2025	102.65	4.334246575	0.67%	99.70146495	4.55%	84.44653148
12	10y	10	1.59%								
13	20y	20	2.15%					**P with YC**	120.7606938		103.9110979
14	30y	30	2.26%					**YTM with YC**	0.62%		
15								**Market YTM**	4.30%		
16								**YC Spread**	3.88%		
17											
18								**TSLA Bid**	103.911		
19											

Pricing TSLA Bond 4/16/21 vs 4/26/21

	A	B	C	D	E	F	G	H
1	**YC rate**				**TSLA**	**Bid prices**	103.911	103.916
2	**t**	**4/16/21**	**4/26/2021**		**Coupons**	**Coupon Amt**	**PV 4/16/21**	**PV 4/26/21**
3	1m	0.02%	0.01%		8/15/2021	2.65	2.65	2.65
4	2m	0.02%	0.02%		2/15/2022	2.65	2.65	2.65
5	3m	0.02%	0.03%		8/15/2022	2.65	2.65	2.65
6	6m	0.04%	0.03%		2/15/2023	2.65	2.64	2.64
7	1y	0.06%	0.07%		8/15/2023	2.65	2.64	2.64
8	2y	0.16%	0.16%		2/15/2024	2.65	2.63	2.63
9	3y	0.34%	0.34%		8/15/2024	2.65	2.61	2.61
10	5y	0.84%	0.83%		2/15/2025	2.65	2.59	2.60
11	7y	1.26%	1.26%		8/15/2025	102.65	99.70	99.78
12	10y	1.59%	1.58%					
13	20y	2.15%	2.14%			**P with YC**	120.76	120.84
14	30y	2.26%	2.25%			**TSLA spread**	3.88%	3.92%
15								

Why might a bond price have increased in 10 days?

Pricing TSLA Bond 4/16/21 vs 4/26/21

	A	B	C	D	E	F	G	H
1	**YC rate**				**TSLA**	**Bid prices**	103.911	103.916
2	**t**	**4/16/21**	**4/26/2021**		**Coupons**	**Coupon Amt**	**PV 4/16/21**	**PV 4/26/21**
3	1m	0.02%	0.01%		8/15/2021	2.65	2.65	2.65
4	2m	0.02%	0.02%		2/15/2022	2.65	2.65	2.65
5	3m	0.02%	0.03%		8/15/2022	2.65	2.65	2.65
6	6m	0.04%	0.03%		2/15/2023	2.65	2.64	2.64
7	1y	0.06%	0.07%		8/15/2023	2.65	2.64	2.64
8	2y	0.16%	0.16%		2/15/2024	2.65	2.63	2.63
9	3y	0.34%	0.34%		8/15/2024	2.65	2.61	2.61
10	5y	0.84%	0.83%		2/15/2025	2.65	2.59	2.60
11	7y	1.26%	1.26%		8/15/2025	102.65	99.70	99.78
12	10y	1.59%	1.58%					
13	20y	2.15%	2.14%			**P with YC**	120.76	120.84
14	30y	2.26%	2.25%			**TSLA spread**	3.88%	3.92%
15								

Why might a bond price have increased in 10 days?

- Bond prices increase between coupon dates with constant IR.
- TSLA excess risk perceived to have increased.
- YC change: overall market risk decreased.

Lecture 37 April 24, 2023

IR Spread and Implied Probability of Default

Setup:

- A is a "risk-free" asset with term structure $s_A(t)$.
- B is a risky asset with term structure $s_B(t)$.

The **implied probability of default** D_t is the (implied) probability such that

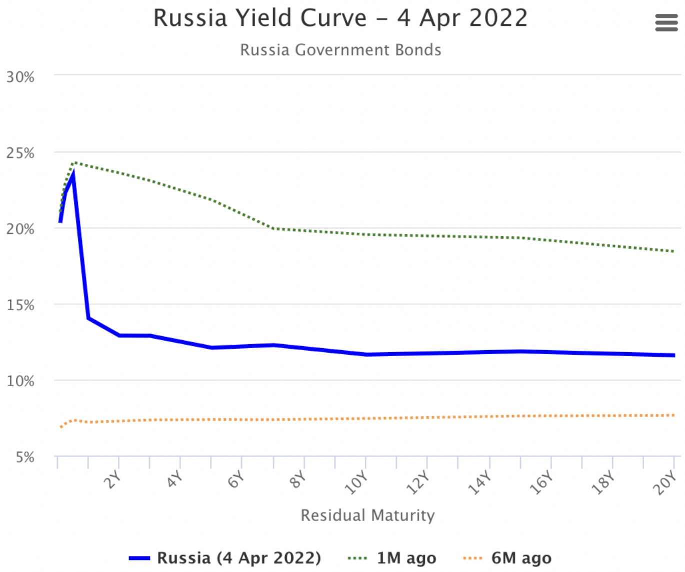
$$\mathrm{PV}_A(\text{cash at time } t) = \mathrm{PV}_B\begin{pmatrix}\text{cash at time } t \text{ with prob} \\ D_t \text{ of getting nothing}\end{pmatrix}.$$

Two options

- Lend money to B
- Lend money to A knowing that with probability D_t, A refuses to repay.

Pricing the Implied Probability of Default

Example: US vs Russia



Example: US vs Russia

$s_{\text{US}}(0.5) = 1.09\%$ vs $s_{\text{Russia}}(0.5) = 23.42\%$.

$$\begin{aligned} D_{0.5}(\text{Russia vs US}) &= 1 - \left(\frac{1 + s_{\text{US}}(0.5)}{1 + s_{\text{Russia}}(0.5)} \right)^{0.5} \\ &= 1 - \left(\frac{1.0109}{1.2342} \right)^{0.5} \\ &= 9.5\%. \end{aligned}$$

TED Spread USD LIBOR vs US 10/08

The **TED Spread** is a spread index between 3-month LIBOR quoted in USD ("Eurodollars") vs 3-month T-Bills. It measures the excess risk of banks borrowing in USD on the international market compared to the US government.

TED Spread USD LIBOR vs US 10/08

On 10/10/08 the TED spread was

$$s_{\mathrm{US}}(0.25) = 0.84\% \qquad s_{\mathrm{LIBOR}}(0.25) = 5.42\% \qquad \mathrm{TED} = 4.58\%$$

What is the implied probability of default for commercial banks borrowing USD on the international markets over the next 3 months?

So what is “risk-free” anyway?

We usually take US Treasury securities to be risk-free.

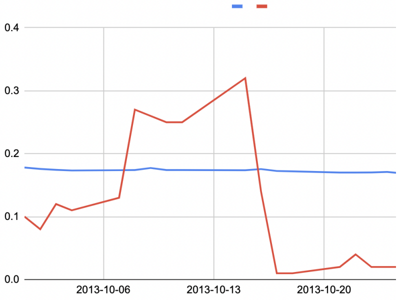


Figure: 1 month Libor USD vs T-Bill

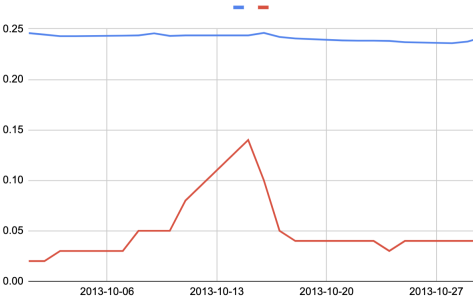


Figure: 3 month Libor USD vs T-Bill