Lecture 29 April 3, 2024

Bonds

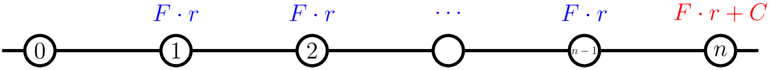
> A **bond** is an instrument which pays periodic **coupon payments** plus a final lump sum **redemption amount** at the end of the term.

Terminology:

- F is the coupon face amount/face value/par value
- r is the coupon rate r is NOT "the interest rate"
- C is the redemption amount, typically the same as the face value
- n is the term of the bond

Bonds

A **bond** is an instrument which pays periodic **coupon payments** plus a final lump sum **redemption amount** at the end of the term.



Questions:

- What should you pay for such a bond?
- Suppose a bond is valued at price P at the moment the bond is issued. What is the implied yield rate?

Bonds typically pay semiannually, but sometimes quarterly. If a bond pays m times a year, it is customary to quote the coupon rate as $r^{(m)}$ and the (implied) yield rate as $i^{(m)}$.

Treasury Auction

TREASURY NEWS

Department of the Treasury • Bureau of the Fiscal Service

Embargoed Until 11:00 A.M.
March 21, 2024

CONTACT: Treasury Auctions
202-504-3550

TREASURY OFFERING ANNOUNCEMENT [1]

Term and Type of Security	2-Year Note
Offering Amount	$66,000,000,000
Currently Outstanding	$0
CUSIP Number	91282CKH3
Auction Date	March 25, 2024
Original Issue Date	April 01, 2024
Issue Date	April 01, 2024
Maturity Date	March 31, 2026
Dated Date	March 31, 2024
Series	AZ-2026
Yield	Determined at Auction

Treasury Results

TREASURY NEWS

Department of the Treasury • Bureau of the Fiscal Service

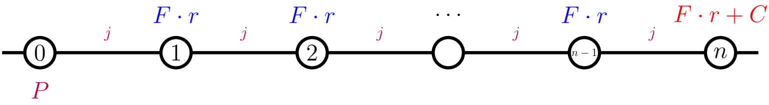
For Immediate Release
March 25, 2024

CONTACT: Treasury Auctions
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	2-Year Note
CUSIP Number	91282CKH3
Series	AZ-2026
Interest Rate	4-1/2%
High Yield [1]	4.595%
Allotted at High	49.16%
Price	99.820388
Accrued Interest per $1,000	$0.12295
Median Yield [2]	4.540%
Low Yield [3]	4.470%
Issue Date	April 01, 2024
Maturity Date	March 31, 2026
Original Issue Date	April 01, 2024
Dated Date	March 31, 2024

Pricing bonds

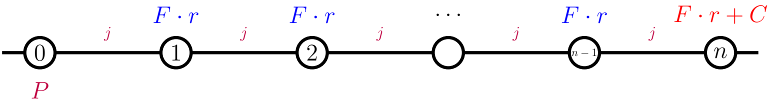


$$P = Fr \cdot a_{\overline{n}|j} + C\nu_j^n$$

$P =$ price at issue

$j =$ implied per period yield rate.

Pricing bonds



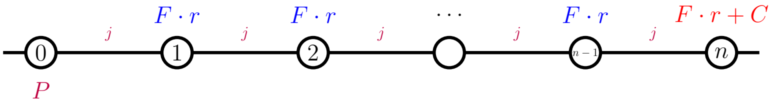
$$P = Fr \cdot a_{\overline{n}|j} + C\nu_j^n$$

$$P = \text{price at issue}$$
$$j = \text{implied per period yield rate.}$$

A **zero coupon bond** is a bond where $r = 0$, i.e., you receive only the redemption.

$$P = C\nu_j^n.$$

Pricing bonds



$$P = Fr \cdot a_{\overline{n}|j} + C\nu_j^n$$

$$P = \text{price at issue}$$
$$j = \text{implied per period yield rate.}$$

High yield price for 3/25/24 auction of 2-year T-Notes.

$$r^{(2)} = 4.5\% \qquad r = 2.25\%$$
$$i^{(2)} = 4.595\% \qquad j = 2.2975\%$$
$$n = 4 \qquad F = C = 100$$
$$P = Fra_{\overline{4}|j} + C\nu_j^4 \qquad P = 99.8204311122389$$
$$\text{real } i^{(2)} = 4.5950228345214\% \quad \text{real } P = 99.820388$$

Yield rate from price

4.1.6

A 25-year bond with a par value of 1000 and 10% coupons payable quarterly is selling at 800. Calculate the annual nominal yield rate convertible quarterly.

$(i^{(4)} = 12.65\%$

Yield rate from price

4.1.6

A 25-year bond with a par value of 1000 and 10% coupons payable quarterly is selling at 800. Calculate the annual nominal yield rate convertible quarterly.

$(i^{(4)} = 12.65\%$

solve 800 = 25*(1-(1+x)^(-100))/x + 1000/(1+x)^100

NATURAL LANGUAGE | MATH INPUT

Input interpretation

solve $800 = 25 \times \frac{1 - \frac{1}{(1+x)^{100}}}{x} + \frac{1000}{(1+x)^{100}}$

Solution over the reals

$x \approx -2.0022$

$x \approx 0.0316179$

Relation between price and yield rate

The results of the auction of 3/25/24 with $F = 100$ and $r^{(2)} = 4.5\%$.

Yield rate	Price
High yield $i^{(2)} = 4.595\%$	99.820388
Par yield $i^{(2)} = r^{(2)} = 4.5\%$	100
Median yield $i^{(2)} = 4.54\%$	99.924342
Low yield $i^{(2)} = 4.47\%$	100.056792

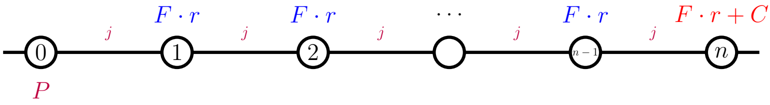
Price vs Yield

As the yield rate goes down the price goes up and vice-versa.

Remark

In a bond sale auction the low yield is the first to occur.

Lecture 30 April 5, 2024



$$P = Fr \cdot a_{\overline{n}|j} + C\nu_j^n$$

- F is par or face value
- r is coupon rate
- $F \cdot r$ is coupon payment
- C is the redemption, typically $C = F$

Pricing bonds: The typical case $F = C$.

$$P = Fr a_{\overline{n}|j} + F\nu_j^n \qquad\qquad P = Fr a_{\overline{n}|j} + F(1 - j a_{\overline{n}|j})$$

$$= Fr\left(\frac{1-\nu_j^n}{j}\right) + F\nu_j^n \qquad\qquad = F + F(r - j)a_{\overline{n}|j}$$

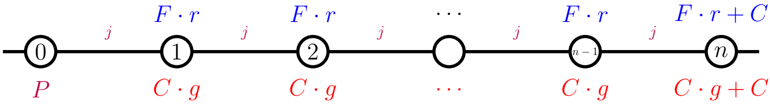
Price vs Yield

As the yield rate j goes down the price P goes up and vice-versa.

	Price	Yield rate
Discount	$P < F$	$j > r$
At par	$P = F$	$j = r$
Premium	$P > F$	$j < r$

Pricing bonds: The general case $F \neq C$.

Algebra trick: use the **modified coupon rate** $g = r \cdot \frac{F}{C}$ with $F \cdot r = C \cdot g$.



$$P = Fra_{\overline{n}|j} + C\nu_j^n = Cg\left(\frac{1-\nu_j^n}{j}\right) + C\nu_j^n$$

$$P = Fra_{\overline{n}|j} + C(1 - ja_{\overline{n}|j}) = C + C(g-j)a_{\overline{n}|j}$$

From now on we will always assume that $F = C$ unless otherwise specified.

Pricing bonds: The general case $F \neq C$.

Algebra trick: use the **modified coupon rate** $g = r \cdot \frac{F}{C}$ with $F \cdot r = C \cdot g$.

$$P = Fr a_{\overline{n}|j} + C\nu_j^n \qquad P = Fr a_{\overline{n}|j} + C(1 - j a_{\overline{n}|j})$$

$$= Cg\left(\frac{1-\nu_j^n}{j}\right) + C\nu_j^n \qquad = C + C(g-j)a_{\overline{n}|j}$$

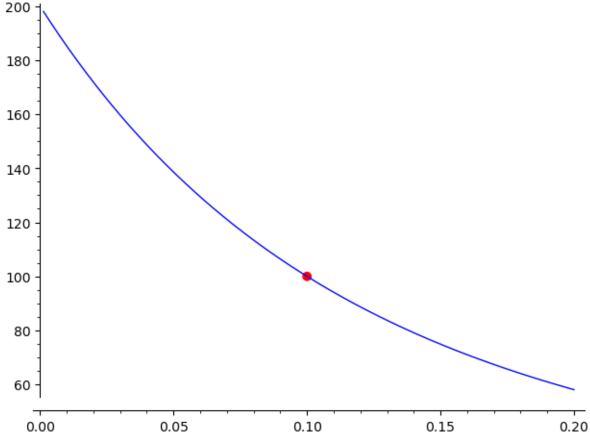
Price vs Yield

As the yield rate j goes down the price P goes up and vice-versa.

	Price	Yield rate
Discount	$P < C$	$j > g$
At par	$P = C$	$j = g$
Premium	$P > C$	$j < g$

Pricing bonds: P vs j

	Price	Yield rate
Discount	$P < F$	$j > r$
At par	$P = F$	$j = r$
Premium	$P > F$	$j < r$



Why would anybody pay more??

The treasury offers 100 10-year bonds of \$100 each, with $r^{(2)} = 2\%$ semiannual nominal coupon rate. In the auction, the only prices offered are \$90, \$100, and \$110.

Price	Nominal yield $i^{(2)}$
\$90	3.17%
\$100	2.00%
\$110	0.95%

Why would anybody pay more??

The treasury offers 100 10-year bonds of $100 each, with $r^{(2)} = 2\%$ semiannual nominal coupon rate. In the auction, the only prices offered are $90, $100, and $110.

Price	Nominal yield $i^{(2)}$
$90	3.17%
$100	2.00%
$110	0.95%

The real choices are

Scenario	Yield
Low yield bond (first to bid)	0.95%
Medium yield bond (middle to bid)	2%
High yield bond (last to bid)	3.17%
No more bonds to bid on	0%
Risky assets	high yield

Exercise

4.1.24

A bond with face and redemption amount of 3000 with *annual* coupons is selling at an effective annual yield rate equal to twice the annual coupon rate. The present value of the coupons is equal to the present value of the redemption amount. What is the selling price?

$(P = 2000)$

Makeham's formula

$$P = \frac{r}{j}F + \left(1 - \frac{r}{j}\right)K = K + \frac{r}{j}(F - K)$$

where K is the PV of the redemption amount.

Pricing bonds with Makeham's formula

TREASURY AUCTION RESULTS

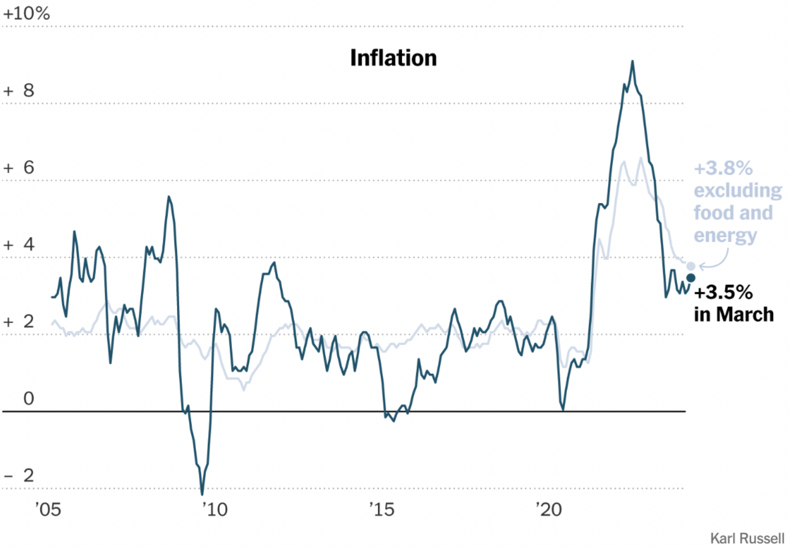
Term and Type of Security	30-Year Bond
CUSIP Number	912810TN8
Series	Bonds of February 2053
Interest Rate	3-5/8%
High Yield [1]	3.686%
Allotted at High	61.77%
Price	98.898317
Accrued Interest per $1,000	None
Median Yield [2]	3.572%
Low Yield [3]	3.500%

$$r^{(2)} = 3.625\% \qquad r = 1.8125\%$$

$$i^{(2)} = 3.686\% \qquad j = 1.843\%$$

$$K = F\nu_j^{60} = 33.42943 \quad P = K + \frac{r}{j}(F - K) = 98.898317.$$

Lecture 32 April 10, 2024

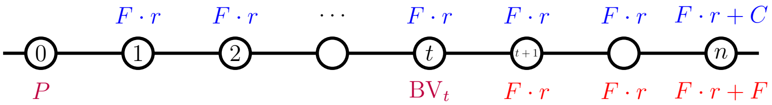


The book value of a bond

Book Value

The **book value** of a bond at time t is the present value of all future coupon and redemption payments using the original purchase yield rate as interest rate.

Immediately after a coupon payment:

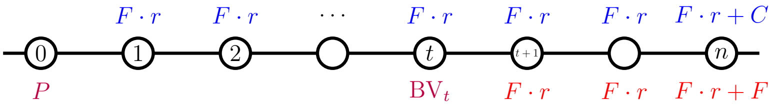


The book value of a bond

Book Value

The **book value** of a bond at time t is the present value of all future coupon and redemption payments using the original purchase yield rate as interest rate.

Immediately after a coupon payment:



BV_t is the price of a bond with the same coupon and yield rate but with term $n - t$:

$$\mathrm{BV}_t = \frac{r}{j} F + \left(1 - \frac{r}{j}\right) K = \frac{r}{j} F + \left(1 - \frac{r}{j}\right) F \nu_j^{n-t},$$

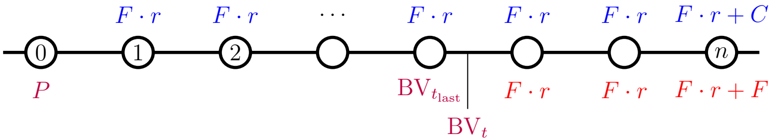
except $\mathrm{BV}_n = 0$.

The book value of a bond

Book Value

The **book value** of a bond at time t is the present value of all future coupon and redemption payments using the original purchase yield rate as interest rate.

Between coupon payments:

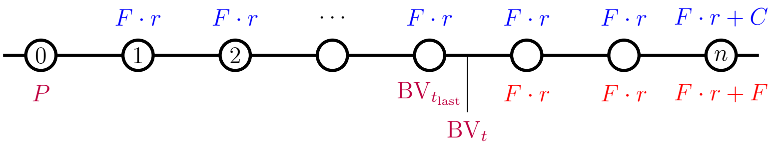


The book value of a bond

Book Value

The **book value** of a bond at time t is the present value of all future coupon and redemption payments using the original purchase yield rate as interest rate.

Between coupon payments:

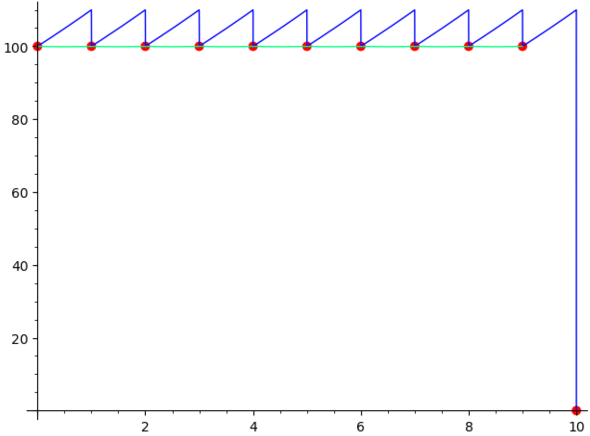


Use time value of money from previous coupon payment! If the previous coupon payment occurred at time t_last then

$$\mathrm{BV}_t = \mathrm{BV}_{t_\mathrm{last}}(1+j)^{t-t_\mathrm{last}}.$$

Book value for at par bonds

$r = 10\%$, $j = 10\%$, $n = 10$



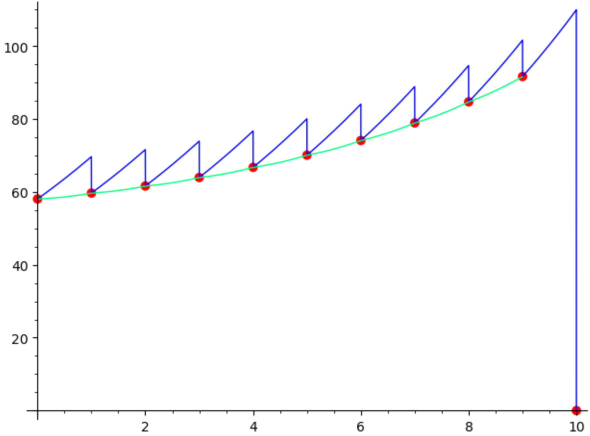
Green graph is called **market price**

$$\text{MP}_t = \text{BV}_t - Fr(t - t_{\text{last}}).$$

Market price stays constant for at par bonds.

Book value for discount bonds

$r = 10\%$, $j = 20\%$, $n = 10$



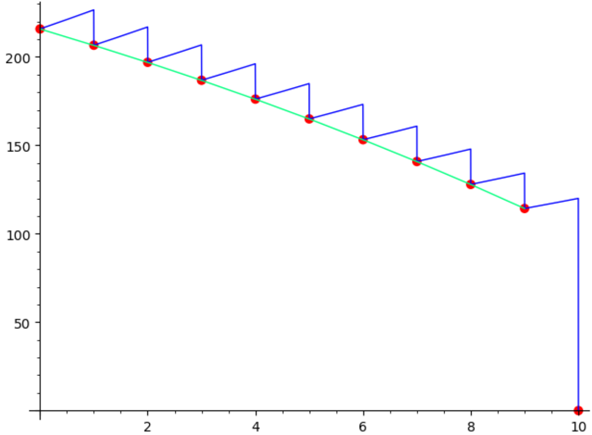
Green graph is called **market price**

$$\mathrm{MP}_t = \mathrm{BV}_t - Fr(t - t_{\text{last}}).$$

Market price stays increases for discount bonds.

Book value for premium bonds

$r = 20\%,\ j = 5\%,\ n = 10$



Green graph is called **market price**

$$\mathrm{MP}_t = \mathrm{BV}_t - Fr(t - t_{\text{last}}).$$

Market price stays decreases for premium bonds.

Amortization for bonds

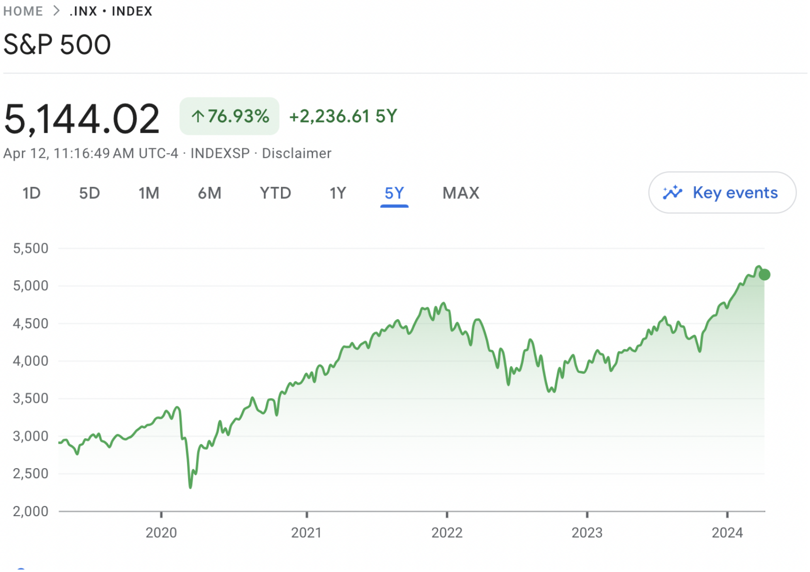
In dealing with bonds the interest rate is the yield rate j, $\mathrm{BV} = \mathrm{OB}$, and PR is called **amount for amortization of principal**.

$$\begin{aligned}
\mathrm{BV}_t &= \text{Price of bond with } r, n-t, F \\
&= Fr\,a_{\overline{n-t}|j} + F\nu_j^{n-t} \\
&= F + F(r-j)a_{\overline{n-t}|j} \\
&= \frac{r}{j}F + \left(1 - \frac{r}{j}\right) F\nu_j^{n-t},
\end{aligned}$$

except $\mathrm{BV}_n = 0$.

- The first formula is easiest to explain.
- The second formula is used in amortization tables.
- The third formula is most effective for understanding the behavior of book value.

Lecture 33 April 12, 2024



Amortization tables for bonds (with $F = C$)

t	I	K	PR	OB/BV
0				$F(1+(r-j)a_{\overline{n}\vert j})$
1	$F(j+(r-j)(1-\nu_j^n))$	Fr	$F(r-j)\nu_j^n$	$F(1+(r-j)a_{\overline{n-1}\vert j})$
2	$F(j+(r-j)(1-\nu_j^{n-1}))$	Fr	$F(r-j)\nu_j^{n-1}$	$F(1+(r-j)a_{\overline{n-2}\vert j})$
		$\vdots$		
k	$F(j+(r-j)(1-\nu_j^{n-k+1}))$	Fr	$F(r-j)\nu_j^{n-k+1}$	$F(1+(r-j)a_{\overline{n-k}\vert j})$
		$\vdots$		
$n-1$	$F(j+(r-j)(1-\nu_j^2))$	Fr	$F(r-j)\nu_j^2$	$F(1+(r-j)a_{\overline{n-1}\vert j})$
n	$F(j+(r-j)(1-\nu_j))$	$Fr+F$	$F(r-j)\nu_j+F$	0

$$\mathrm{BV}_t = F + F(r-j)a_{\overline{n-t}|j}.$$

Amortization tables for bonds (with $F = C$)

t	I	K	PR	OB/BV
0				$F(1+(r-j)a_{\overline{n}\vert j})$
1	$F(j+(r-j)(1-\nu_j^n))$	Fr	$F(r-j)\nu_j^n$	$F(1+(r-j)a_{\overline{n-1}\vert j})$
2	$F(j+(r-j)(1-\nu_j^{n-1}))$	Fr	$F(r-j)\nu_j^{n-1}$	$F(1+(r-j)a_{\overline{n-2}\vert j})$
		$\vdots$		
k	$F(j+(r-j)(1-\nu_j^{n-k+1}))$	Fr	$F(r-j)\nu_j^{n-k+1}$	$F(1+(r-j)a_{\overline{n-k}\vert j})$
		$\vdots$		
$n-1$	$F(j+(r-j)(1-\nu_j^2))$	Fr	$F(r-j)\nu_j^2$	$F(1+(r-j)a_{\overline{n-1}\vert j})$
n	$F(j+(r-j)(1-\nu_j))$	$Fr+F$	$F(r-j)\nu_j+F$	0

$$\mathrm{BV}_t = F + F(r-j)a_{\overline{n-t}|j}.$$

PR

The principal repaid is **positive** for premium bonds, but **negative** for discount bonds.

Exercise

4.2.7

Among a company's assets and accounting records, an actuary finds a 15-year bond that was purchased at a premium. From the records, the actuary has determined the following:

(i) The bond pays semi-annual interest.

(ii) The amount for amortization of the premium in the 2^{nd} coupon payment was 977.19.

(iii) The amount for amortization of the premium in the 4^{th} coupon payment was 1046.79.

What is the value of the premium?

($j = 3.5\%$, 48739.29. Here **premium** means $P - F$, i.e., what is paid above the par value.)

Callable Bonds

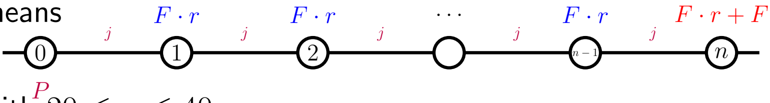
Callable Bond

A **callable bond** is a bond with par value F and coupon rate r for which the term n is not fixed, and can be decided ("called") by the bond issuer. Typically, the term is chosen in an interval specified contractually.

For example, British perpetuities (consols) that we discussed earlier were callable by an act of Parliament.

Example

A bond with $r^{(2)} = 5\%$ callable between 10 and 20 years means



with $20 \le n \le 40$.

Callable Bonds

Big Question: How do you compute the price of a callable bond

$$P = F + F(r - j)a_{\overline{n}|j}$$

if you, the buyer, don't have a say in the choice of n?

Callable Bonds

Big Question: How do you compute the price of a callable bond

$$P = F + F(r - j)a_{\overline{n}|j}$$

if you, the buyer, don't have a say in the choice of n?

Correct Question 1

What price should you pay for a callable bond if you want to guarantee a certain yield rate? (If you pay too much, the yield rate might be too low.)

Correct Question 2

What yield rate are you guaranteed to get if you pay a certain price for a callable bond?

Yield rate vs term for callable bonds

A callable bond pays semiannually with $r^{(2)} = 10\%$, and can be called any time between years 15 and 20.

$P = 80$		$P = 100$		$P = 120$	
n	j	n	j	n	j
31	6.518%	31	5%	31	3.880%
32	6.500%	32	5%	32	3.896%
33	6.483%	33	5%	33	3.911%
34	6.468%	34	5%	34	3.925%
35	6.454%	35	5%	35	3.938%
36	6.440%	36	5%	36	3.950%
37	6.428%	37	5%	37	3.961%
38	6.417%	38	5%	38	3.972%
39	6.406%	39	5%	39	3.982%
40	6.396%	40	5%	40	3.991%

Yield rate vs term for callable bonds

A callable bond pays semiannually with $r^{(2)} = 10\%$, and can be called any time between years 15 and 20.

$P = 80$		$P = 100$		$P = 120$	
n	j	n	j	n	j
31	6.518%	31	5%	31	3.880%
32	6.500%	32	5%	32	3.896%
33	6.483%	33	5%	33	3.911%
34	6.468%	34	5%	34	3.925%
35	6.454%	35	5%	35	3.938%
36	6.440%	36	5%	36	3.950%
37	6.428%	37	5%	37	3.961%
38	6.417%	38	5%	38	3.972%
39	6.406%	39	5%	39	3.982%
40	6.396%	40	5%	40	3.991%
	5%		5%		5%

Yield rate vs term for callable bonds

Theorem

Suppose a callable bond has face value F, coupon rate r, callable at any time $a \leq n \leq b$.

1. If the bond sells at a discount then the yield rate j is largest early $n = a$ and lowest late when $n = b$.
2. If the bond sells at a premium then the yield rate j is lowest early $n = a$ and largest late when $n = b$.

Type of bond	Term to use for calculating price
Discount	Latest possible call date
Premium	Earliest possible call date

Exercise

4.3.1

A 10% bond with face amount 100 is callable on any coupon date from 15½ years after issue up to the maturity date which is 20 years from issue.

(a) Find the price of the bond to yield a minimum nominal annual rate of (i) 12%, (ii) 10%, and (iii) 8%.

(b) Find the minimum annual yield to maturity if the bond is purchased for (i) 80, (ii) 100, and (iii) 120.

((a) 84.9537, 117.5885 (b) 12.79%, 7.76%)