Lecture 34 April 15, 2024

Chapter	Topic
1	Time value of money
2	Annuities
3	Loans
4	Bonds
5	Measuring an investment
6	Varying interest rates
7	Hedging against interest rate changes

How to value investments?

How do you measure how well an investment is doing?

- (Effective annual) yield rate.
  PRO:

  CON:

- Implied (effective annual) interest rate.
  PRO:

  CON:

- This chapter: Net present value, time and dollar weighted returns.

How to value investments?

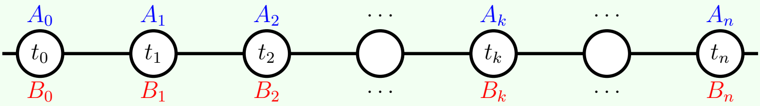
How do you measure how well an investment is doing?

- (Effective annual) yield rate.
  PRO: Straightforward to compute
  Conceptually easy to explain
  CON: Can't account for investing money at different times
- Implied (effective annual) interest rate.
  PRO: Works for bonds and investing at different times
  CON: Might not exist or might be meaningless
- This chapter: Net present value, time and dollar weighted returns.

What is an investment?

Investment

By an investment we mean a series of cashflows



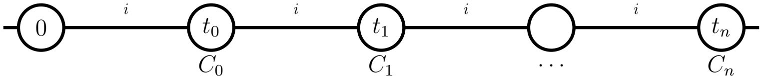
- Receive A_k at time t_k
- Pay B_k at time t_k
- Cash flow at time t_k is $C_k = A_k - B_k$.

Internal Rate of Return

The IRR of an investment is the implied effective annual interest rate i such that

$$\mathrm{PV}_i(C_0, C_1, \ldots, C_n) = 0.$$

IRR Equation of Value



IRR

The IRR is any effective annual interest rate i such that

$$C_0\nu_i^{t_0} + C_1\nu_i^{t_1} + \cdots + C_n\nu_i^{t_n} = 0$$

$$\frac{C_0}{(1+i)^{t_0}} + \frac{C_1}{(1+i)^{t_1}} + \cdots + \frac{C_n}{(1+i)^{t_n}} = 0.$$

Examples

Loans:

- Loan amount L at time $t_0 = 0$.
- Payments $K_1, \ldots, K_n$ at times $t_1 = 1, \ldots, t_n = n$.
- Cash flow $C_0 = L$, $C_1 = -K_1, \ldots, C_n = -K_n$.

Bonds:

- Price P.
- Coupons $F \cdot r, \ldots, F \cdot r$, $F \cdot r + F$.
- Cash flow $C_0 = -P$, $C_1 = F \cdot r, \ldots,$ $C_{n-1} = F \cdot r$, $C_n = F \cdot r + F$.

Exercise

5.1.7

Smith buys an investment property for 900,000 by making a down payment of 150,000 and taking a loan for 750,000. Starting one month after the loan is made Smith must make monthly loan payments, but he also receives monthly rental payments, set for 2 years such that his net outlay per month is 1200. In addition there are taxes of 10,000 payable 6 months after the loan is made and annually thereafter as long as Smith owns the property. Two years after the original purchase date Smith sells the property for $Y \geq 741,200$, out of which he must pay the balance of 741,200 on the loan.

1. What is the IRR if $Y = 1m$?
2. What is the smallest Y for which the IRR is positive?

(IRR is 16.39% effective annual, $i^{(12)} = 15.276\%$ nominal annual, $Y \geq 938800$)

Exercise

You always start with a timeline and a cashflow

$$
\begin{aligned}
C_0 &= -150k \\
C_k &= -1200 \qquad k \neq 6, 18, 24 \\
C_6 &= -11200 \\
C_{18} &= -11200 \\
C_{24} &= Y - 741200
\end{aligned}
$$

Exercise

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(2): The bigger the sale price Y, the bigger the IRR. To find we smallest Y for which IRR is positive, we can preted IRR is 0.

Exercise

You always start with a timeline and a cashflow

$$
\begin{aligned}
C_0 &= -150k \\
C_k &= -1200 \qquad k \neq 6, 18, 24 \\
C_6 &= -11200 \\
C_{18} &= -11200 \\
C_{24} &= Y - 741200
\end{aligned}
$$

We have to solve

$$-150000 - 1200 a_{\overline{24}|j} - 10000(\nu^6 + \nu^{18}) + (Y - 741200)\nu^{24} = 0.$$

Solving the equation

Input interpretation

solve	$-150000 - 1200 \times \frac{1 - \frac{1}{(1+x)^{24}}}{x} - 10000\left(\frac{1}{(1+x)^6} + \frac{1}{(1+x)^{18}}\right) + \frac{1000000 - 741200}{(1+x)^{24}} = 0$

Results

More roots

$x \approx -2.01870$

$x \approx 0.0125132$

$x \approx -1.98803 - 0.26535\,i$

$x \approx -1.98803 + 0.26535\,i$

$x \approx -1.88982 - 0.51371\,i$

Problems with IRR

We define the IRR as an implied interest rate such that

$$\mathrm{PV}_i(C_0, C_1, \ldots, C_n) = 0.$$

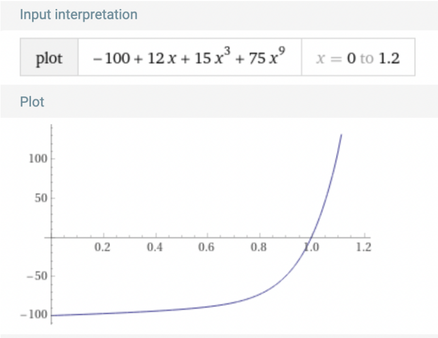
Possible problems:

- A solution for i might not exist.
- More than one solution for i might exist.
- A solution obtained might be meaningless.

Unique solution to IRR equation

Theorem

Suppose an investment has $C_0 = L > 0$ (resp. $L < 0$) and $C_1, \ldots, C_n < 0$ (resp. $C_1, \ldots, C_n > 0$). Then there is a unique IRR.



Problems with IRR: no solution

Example: You sell a 10-year bond with par value 100, annual coupon rate 5%, for 110 and buy a 20-year bond with par value 100, annual coupon rate 10%, for 90. What is the IRR?

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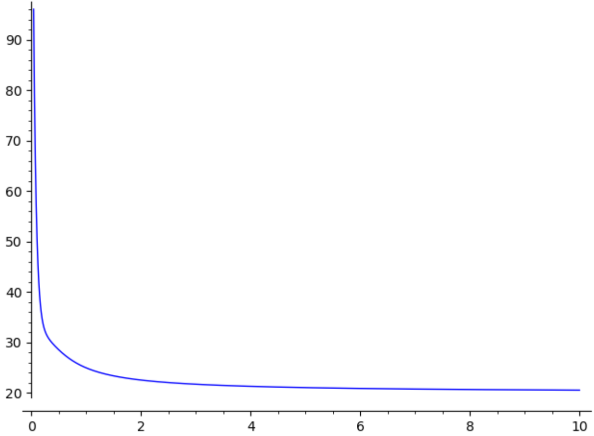

Problems with IRR: ineffective in comparing investments

5.1.4

Transactions A and B are to be compared. Transaction A has net cashflows of

$$C_0^A = -5, \; C_1^A = 3.72, \quad C_2^A = 0, \quad C_3^A = 4$$

and Transaction B has net cashflows

$$C_0^B = -5, \; C_1^B = 3, \; C_2^B = 1.7, \; C_3^B = 3.$$

Question: What are the IRR for A and B? ($\nu_A = 0.797891$, $i_A = 25.330403\%$ and $\nu_B = 0.797906$, $i_B = 25.32804\%$)

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IRR

IRR is the annual interest rate i that solves

$$\mathrm{PV}_i(C_0, \ldots) = 0$$

Problems with IRR: multiple solutions

5.1.4

Transactions A and B are to be compared. Transaction A has net cashflows of

$$C_0^A = -5, \ C_1^A = 3.72, \quad C_2^A = 0, \quad C_3^A = 4$$

and Transaction B has net cashflows

$$C_0^B = -5, \ C_1^B = 3, \ C_2^B = 1.7, \ C_3^B = 3.$$

Question: You sell A and buy B. What is the IRR?
$(11.11\%, 25\%)$

Net Present Value

The IRR frequently can't compare investment opportunities.

Better Question

Instead of asking "what interest rate" each investment yields ask "which investment is more advantageous depending on prevailing market interest rates"?

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The **Net Present Value** of an investment is simply the present value PV_i at a certain (market) interest rate i, which can be thought of as **cost of capital**.

NPV in comparing investments

5.1.4

Transactions A and B are to be compared. Transaction A has net cashflows of

$$C_0^A = -5, \ C_1^A = 3.72, \quad C_2^A = 0, \quad C_3^A = 4$$

and Transaction B has net cashflows

$$C_0^B = -5, \ C_1^B = 3, \ C_2^B = 1.7, \ C_3^B = 3.$$

Question: For what (market prevailing) interest rates i is A better than B? ($i < 11.11\%$ or $i > 25\%$)

Continuous versions of IRR and NPV

	IRR	NPV
Idea	$\mathrm{PV}_i(\mathrm{Cashflows}) = 0$	$= \mathrm{PV}_i(\mathrm{Cashflows})$
Discrete	$C_0 + \frac{C_1}{1+i} + \cdots + \frac{C_n}{(1+i)^n} = 0$	$C_0 + \frac{C_1}{1+i} + \cdots + \frac{C_n}{(1+i)^n}$
Continuous	$\int_0^n C_t e^{-\delta t} dt = 0$	$\int_0^n C_t e^{-\int_0^t \delta_x dx} dt.$

Exercise

5.1.11

Suppose a company is marketing a new product. The production and marketing process involves a startup cost of 1,000,000 and continuing cost of 200,000 per year for 5 years, paid continuously. It is forecast that revenue from the product will begin one year after startup, and will continue until the end of the original 5-year production process. Revenue (which will be received continuously) is estimated to start at a rate of 500,000 per year and increase linearly (and continuously) over a two-year period to a rate of 1,000,000 per year at the end of the 3^{rd} year, and then decrease to a rate of 200,000 per year at the end of the 5^{th} year. Solve for the yield rate δ earned by the company over the 5-year period.

$(\delta = 17.42\%)$

Exercise

-1000+250*[integral from 1 to 3 of (1+x)*e^(-xy)dx]-200*[integral from 0 to 5 of e^(-xy)dx]+400*[integral fr

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD | EXAMPLES | UPLOAD | RANDOM

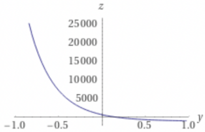
Input

$$-1000 + 250\int_1^3 (1+x)\,e^{-x\,y}\,dx - 200\int_0^5 e^{-x\,y}\,dx + 400\int_3^5 (5.5-x)\,e^{-x\,y}\,dx$$

Result

$$\frac{250\,e^{-3\,y}\left(-4\,y + e^{2\,y}\,(2\,y+1) - 1\right)}{y^2} + \frac{400\,e^{-5\,y}\left(-0.5\,y + e^{2\,y}\,(2.5\,y-1) + 1\right)}{y^2} - \frac{200\left(1-e^{-5\,y}\right)}{y} - 1000$$

Plots

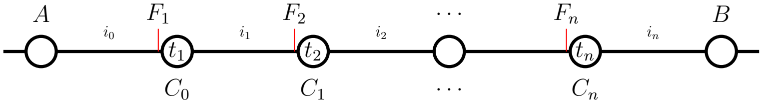


(y from −1 to 1)

Time weighted IRR

Idea

Given an investment we'd like a **yield** that is truly internal, i.e., one that isn't affected by cashflows.



Ignoring cash flows the investment behaves as follows:

Period	Start	End	Per period yield
0	A	F_1	i_0
1	$F_1 + C_1$	F_2	i_2
2	$F_2 + C_2$	F_3	i_3
$\vdots$			
n	$F_n + C_n$	B	i_n

Time weighted IRR

Definition

The **time weighted IRR** is the average rate of the in-between cash flows rates of return:

$$(1+i)^N = (1+i_0)(1+i_1)\cdots(1+i_n) = \frac{F_1}{A}\cdot\frac{F_2}{F_1+C_1}\cdot\frac{F_3}{F_2+C_2}\cdots\frac{B}{F_n+C_n}.$$

Example

Suppose we start with 100 and at the end of every year, for 10 years, we deposit 100 into an account at interest rate 5%. What are the IRR and time-weighted IRR? (6.657% and 5%)

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Recall from last time

Definition

The **time weighted IRR** is the average rate of the in-between cash flows rates of return:

$$(1+i)^N = (1+i_0)(1+i_1)\cdots(1+i_n) = \frac{F_1}{A}\cdot\frac{F_2}{F_1+C_1}\cdot\frac{F_3}{F_2+C_2}\cdots\frac{B}{F_n+C_n}.$$

Exercise

5.2.1

The details regarding fund value, contributions and withdrawals from a fund are as follows:

	Date	Amount
Fund Values:	1/1/15	1,000,000
	7/1/15	1,310,000
	1/1/16	1,265,000
	7/1/16	1,540,000
	1/1/17	1,420,000
Contributions Received:	6/30/15	250,000
	6/30/16	250,000
Benefits Paid:	12/31/15	150,000
	12/31/16	150,000

Find the effective annual time-weighted rate of return for the two-year period of 2015 and 2016.

(9.1%)

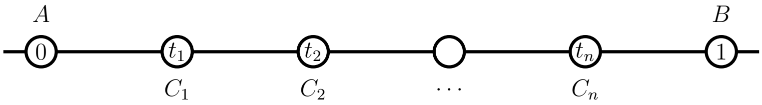
Exercise

Date	Fund	Activity
1/1/15	1m	
6/30/15		250k
7/1/15	1.31m	
12/31/15		-150k
1/1/16	1.265m	
6/30/16		250k
7/1/16	1.54m	
12/31/16		-150k
1/1/17	1.42m	

Dollar-weighted IRR for 1 year

Idea

Pretend money in the investment earns **simple interest** over the course of the year.



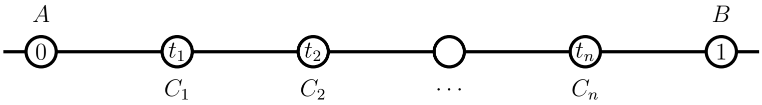
Face value of money earned:

$$I = B - (A + C_1 + C_2 + \cdots + C_n)$$

Dollar-weighted IRR for 1 year

Idea

Pretend money in the investment earns **simple interest** over the course of the year.



Face value of money earned:

$$I = B - (A + C_1 + C_2 + \cdots + C_n)$$

Simple interest money earned:

$$\text{SI} = Ai + C_1 i(1 - t_1) + C_2 i(1 - t_2) + \cdots + C_n i(1 - t_n)$$

Dollar-weighted IRR for 1 year

The **dollar-weighted IRR** is the implied simple interest obtained by equating the "interest" earned I with the simple interest earned SI:

$$i = \frac{B - (A + C_1 + C_2 + \cdots + C_n)}{A + C_1(1 - t_1) + C_2(1 - t_2) + \cdots + C_n(1 - t_n)}.$$

Exercise

5.2.2

You are given the following information about an investment account:

Date	Value Immediately Before Deposit	Deposit
January 1	10	
July 1	12	X
December 31	X	

Over the year, the time-weighted return is 0%, and the dollar-weighted return is Y. Calculate Y.

(-25%)