The Term Structure of Interest Rates

What affects interest rates?

- Risk for a specific investment.
- Collaterals.
- Prevailing cost of capital.
- Length of time.

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Definition

A **term structure** is a relationship between the term of a loan and its interest rate.

The US Treasury Yield Curve

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
03/01/2022	0.11	0.21	0.32	0.60	0.91	1.31	1.47	1.56	1.67	1.72	2.19	2.11
03/02/2022	0.13	0.24	0.34	0.68	1.06	1.50	1.67	1.74	1.83	1.86	2.32	2.24
03/03/2022	0.19	0.25	0.38	0.69	1.08	1.53	1.69	1.74	1.82	1.86	2.32	2.24
03/04/2022	0.15	0.21	0.34	0.69	1.05	1.50	1.62	1.65	1.70	1.74	2.23	2.16
03/07/2022	0.17	0.23	0.38	0.75	1.07	1.55	1.68	1.71	1.77	1.78	2.29	2.19

Date	1 Mo	2 Mo	3 Mo	4 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
04/03/2023	4.70	4.79	4.90	4.98	4.88	4.60	3.97	3.73	3.52	3.48	3.43	3.78	3.64
04/04/2023	4.66	4.80	4.88	4.90	4.80	4.50	3.84	3.60	3.39	3.38	3.35	3.72	3.60
04/05/2023	4.62	4.77	4.86	4.90	4.82	4.43	3.79	3.55	3.36	3.34	3.30	3.67	3.56
04/06/2023	4.57	4.85	4.91	4.98	4.93	4.51	3.82	3.59	3.37	3.34	3.30	3.66	3.54
04/07/2023	4.56	4.90	4.95	5.07	4.95	4.61	3.97	3.72	3.49	3.45	3.39	3.73	3.61

The US Treasury Yield Curve

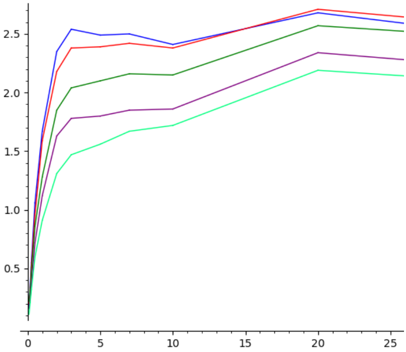


Figure: YC 3/22

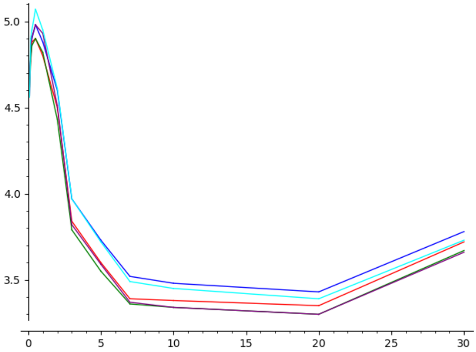


Figure: YC 4/23

Yield Curves vs Credit Rating

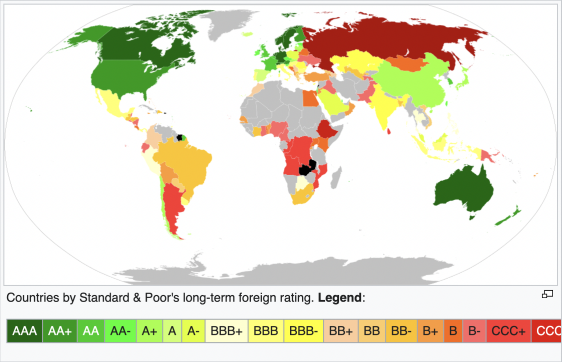


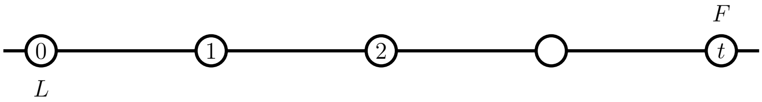
Figure: S&P Credit Rating, Wikipedia 3/22

Lecture 37 April 22, 2024

Term Structure as a Function

Definition

We denote r_t the effective annual interest rate for borrowing money at time 0 to be repaid at time t.



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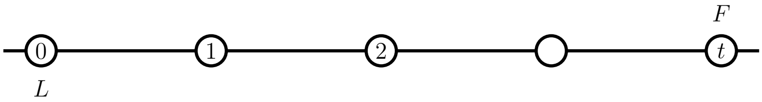
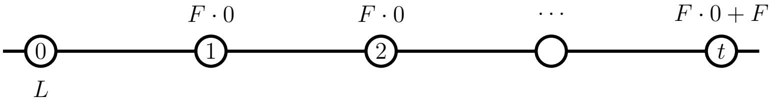


Remark

A loan to be repaid in full at term is a 0-coupon bond.

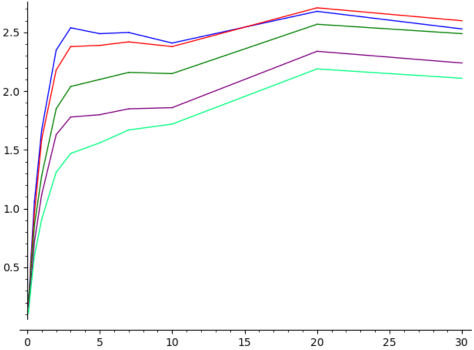


r_t is the yield rate of a 0-coupon bond with term t.

Term structure of spot rate and the Yield Curve

The term structure of interest rates is the collection r_t of 0-coupon yield rates. The yield rates r_t are called **spot rates of interest**.

The term structure depends on when the present, i.e., $t = 0$, occurs, and it typically changes from day to day.



Pricing Assets with the Yield Curve

If interest rates are smaller for short term loans than for long term loans, we should discount future money more if farther in the future.

- 0-coupon bond with term t

- Accumulation factor at time t

- Net Present Value

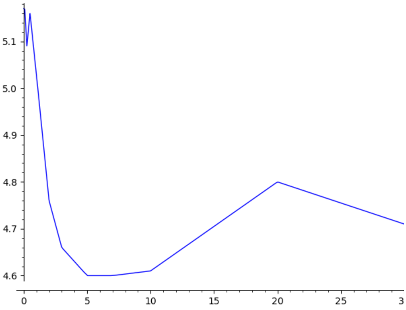


Figure: YC 11/13/06

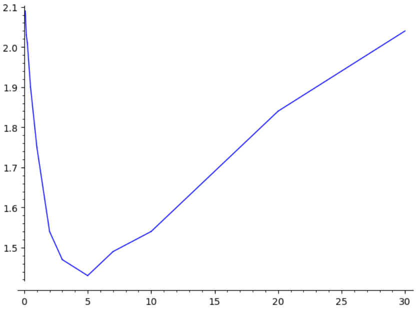
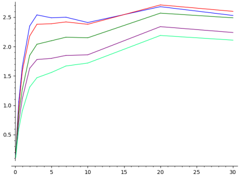


Figure: YC 8/26/19

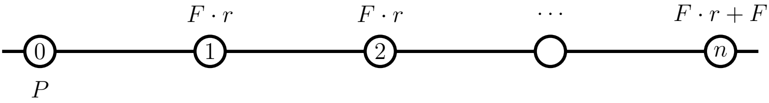
Usual YC (March 2022)



Pricing Bonds with the Yield Curve

Example

Suppose we have a standard bond with F, r, n. If the term structure is (r_t), how should we price this bond?



Exercise

6.1.1

You are given the following term structure:

$$r_1 = .15, \qquad r_2 = .10, \qquad r_3 = .05.$$

These are *effective annual rates of interest* for zero coupon bonds of 1, 2 and 3 years maturity, respectively. A newly issued 3-year bond with face amount 100 has annual coupon rate 10%, with coupons paid *once per year* starting one year from now.

Find the price and effective annual yield to maturity of the bond.

(111.98, 5.56%)

Term Structure

Interest rate depends on the **term** with r_t the annual interest rate for a loan taken now and due at time t.

$$a(t) = (1 + \underbrace{i}_{r_t})^t$$

The term structure enters computations ONLY via $a(t)$.

Bond example

For Immediate Release
March 27, 2023

CONTACT: Treasury Auctions
202-504-3550

TREASURY AUCTION RESULTS

Term and Type of Security	2-Year Note
CUSIP Number	91282CGU9
Series	AZ-2025
Interest Rate	3-7/8%
High Yield [1]	3.954%
Allotted at High	13.09%
Price	99.849511
Accrued Interest per $1,000	None
Median Yield [2]	3.870%
Low Yield [3]	3.800%
Issue Date	March 31, 2023

Type	Yield	Price
High yield/Low price	3.954%	99.849511
Median yield/Median price	3.870%	100.009534
Low yield/High price	3.800%	100.143137

Bond example

Yield curve on 3/27/2023

1m	2m	3m	4m	6m	1y	2y
4.22	4.47	4.91	4.90	4.86	4.51	3.94

3y	5y	7y	10y	20y	30y
3.79	3.59	3.57	3.53	3.90	3.77

Computed price of 3/27/2023 bond:

$$P = 99.922638$$

Lecture 38 April 24, 2024

Constructing the Yield Curve

If we know the term structure (r_t) for $t \leq n$ we can price annual bonds with term n:

$$\begin{aligned} P_{F,r,n} &= \frac{Fr}{a(1)} + \frac{Fr}{a(2)^2} + \cdots + \frac{Fr + F}{a(n)^n} \\ &= \frac{Fr}{1 + r_1} + \frac{Fr}{(1 + r_2)^2} + \cdots + \frac{Fr + F}{(1 + r_n)^n} \end{aligned}$$

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Yield Curve from Bonds

If we looked at the prices of many bonds of different kinds of terms, we could obtain many price equations and **deduce** the term structure of interest rates, in other words we can construct the yield curve.

Exercise

6.1.5

You are given the following information for 4 bonds. All coupon and yield-to-maturity rates are nominal annual convertible twice per year.

Bond	Time to Maturity	Coupon Rate	YTM
1	½-year	4%	.05
2	1-year	6%	.10
3	1½-year	4%	.15
4	2-year	8%	.15

Find the associated term structure for zero coupon bonds with maturities of ½-year, 1-year, 1½-year, and 2-year (quotations should be nominal annual rates convertible twice per year).

(Prices 99.512195, 96.281179, 85.697108, 88.277358, Spot rates 5%, 10.078%, 15.151%, 15.23%)

Exercise

Bond price $P = Fra_{\overline{n}|i} + F\nu_i^n$ for YTM and

$$P_{F,r,n} = \frac{Fr}{a(1)} + \frac{Fr}{a(2)^2} + \cdots + \frac{Fr+F}{a(n)^n}$$
$$= \frac{Fr}{1+r_1} + \frac{Fr}{(1+r_2)^2} + \cdots + \frac{Fr+F}{(1+r_n)^n}$$

using the yield curve.

Bond	Term	n	$r^{(2)}$	r	$i^{(2)}$	i	P
1	0.5	1	4%	2%	5%	2.5%	99.512195
2	1	2	6%	3%	10%	5%	96.281179
3	1.5	3	4%	2%	15%	7.5%	85.697108
4	2	4	8%	4%	15%	7.5%	88.277358

Exercise

$$
\begin{aligned}
P_1 &= 99.512195 = \frac{102}{\color{red}{a(0.5)}} \\
a(0.5) &= 1.025 \\
P_2 &= 96.281179 = \frac{3}{a(0.5)} + \frac{103}{\color{red}{a(1)}} \\
a(1) &= 1.103323 \\
P_3 &= 85.697108 = \frac{2}{a(0.5)} + \frac{2}{a(1)} + \frac{102}{\color{red}{a(1.5)}} \\
a(1.5) &= 1.245 \\
P_4 &= 88.277358 = \frac{4}{a(0.5)} + \frac{4}{a(1)} + \frac{4}{a(1.5)} + \frac{104}{\color{red}{a(2)}} \\
a(2) &= 1.3413
\end{aligned}
$$

Exercise

$$
\begin{aligned}
a(0.5) &= 1.025 = 1 + r_{0.5} \\
r_{0.5} &= 2.5\% \\
r_{0.5}^{(2)} &= 5\% \\
a(1) &= 1.103323 = (1 + r_1)^2 \\
r_1 &= 5.04\% \\
r_1^{(2)} &= 10.08\% \\
a(1.5) &= 1.245 = (1 + r_{1.5})^3 \\
r_{1.5} &= 7.58\% \\
r_{1.5}^{(2)} &= 15.16\% \\
a(2) &= 1.3413 \\
r_2 &= 7.62\% \\
r_2^{(2)} &= 15.24\%
\end{aligned}
$$

Forward Interest Rates

Suppose we are given a yield curve/term structure (r_t). The **forward interest rate** between t_1 and t_2 is the implied constant interest rate $i(t_1, t_2)$ for a loan between t_1 and t_2.

Implied by what? By

$$a(t_1)a(t_1, t_2) = a(t_2)$$

Forward Interest Rates

Suppose we are given a yield curve/term structure (r_t). The **forward interest rate** between t_1 and t_2 is the implied constant interest rate $i(t_1, t_2)$ for a loan between t_1 and t_2.

Implied by what? By

$$a(t_1)a(t_1, t_2) = a(t_2)$$
$$(1 + r_{t_1})^{t_1}(1 + i(t_1, t_2))^{t_2 - t_1} = (1 + r_{t_2})^{t_2}$$

Exercise

6.3.5

The following term structure is given as effective annual rates of interest on zero coupon bonds:

1-year maturity: 6% 2-year maturity: 7% 3-year maturity: 9%

(a) Find (i) the 1-year forward effective annual interest rate for a 1-year period, $f_{[1,2]}$ and (ii) the 2-year forward effective annual interest rate for a 1-year period, $f_{[2,3]}$.

(b) The effective annual rate of interest for a 4-year zero coupon bond is r_4. Find the minimum value of r_4 needed so that $f_{[3,4]} \geq f_{[2,3]}$, where $f_{[3,4]}$ is the 3-year forward effective annual interest rate for a 1-year period and $f_{[2,3]}$. is found in part (a).

$(8.01\%$, 13.11%, and $\geq 10.01\%)$

Exercise

The term structure of interest rates in the capital markets is currently the following

0.5	5%
1	7%
1.5	9%
2	11%
2.5	9%
≥ 3	7%

What is the 1-year forward yield curve?

1-year forward 0.5	
1-year forward 1	
1-year forward 1.5	
1-year forward 2	
1-year forward 2.5	
1-year forward ≥ 3	

Exercise

The term structure of interest rates in the capital markets is currently the following

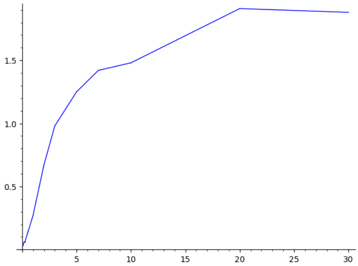
0.5	5%
1	7%
1.5	9%
2	11%
2.5	9%
≥ 3	7%

What is the 1-year forward yield curve?

1-year forward 0.5	13.11%
1-year forward 1	15.15%
1-year forward 1.5	10.35%
1-year forward 2	7%
1-year forward 2.5	7%
1-year forward ≥ 3	7%

Yield Curve as a function from 0 to ∞

Suppose you are quoted a term structure/yield curve (r_t) at a discrete set of points? How do you interpolate in between these dates?



A convenient way is to interpolate linearly.

Dividend Discount Model: Take 2

The DDM prices a stock as the PV of all future dividend payments $P = \sum_{t=1}^{\infty} \mathrm{PV}(d_t)$.

In a world where investors are **risk neutral**, i.e., they don't care about the extra risk of investing in stock compared to the safety of the US Treasury yield curve r_t:

$$P = \sum_{t=1}^{\infty} \frac{d_t}{(1 + r_t)^t}.$$

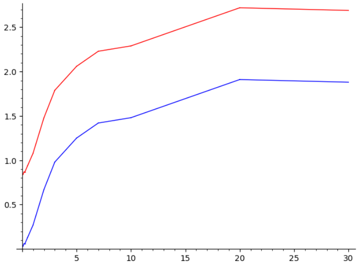
We've already seen that Apple is charged for excess risk!

Apple Stock Price on 2/16/2023

The discrepancy between the price of stock and the DDM price can be interpreted as saying that the market assesses an **excess risk penalty** for Apple vs US.

Let's compute this **excess risk** as the value of x such that term structure $s_W(t)$ that we use to compute PV of Walmart dividends is a **parallel shift** of the US Treasury yield curve:

$$s_W(t) = r_t + x, \text{ for all } t.$$



Apple Stock Price on 2/16/2023

1		Dividend	YC IR	Years	PV		Variables	
2	2023-02-16	0.92	---	---	---		r	8.76%
3	2023-05-16	1.00	4.84%	0.24	0.9719627769			
4	2023-08-16	1.00	4.98%	0.50	0.9432239336		**Computed**	
5	2023-11-16	1.00	4.99%	0.75	0.9155823426			
6	2024-02-16	1.00	4.99%	1.00	0.8887308923		**Apple Price**	
7	2024-05-16	1.09	4.90%	1.25	0.9398426204		2023-02-16	153.71
8	2024-08-16	1.09	4.81%	1.50	0.9136203473			
9	2024-11-16	1.09	4.71%	1.75	0.888504261		**DDM**	
10	2025-02-16	1.09	4.62%	2.00	0.8644387103		i	9.41%
11	2025-05-16	1.18	4.55%	2.25	0.9154781364		Price	153.8461538
12	2025-08-16	1.18	4.49%	2.50	0.890872271			
13	2025-11-16	1.18	4.42%	2.75	0.8671919305		**DDM YC**	
14	2026-02-16	1.18	4.35%	3.00	0.8443985062		Excess Risk x	7.53%
15	2026-05-16	1.29	4.31%	3.25	0.8945116168		Price	153.7059725
16	2026-08-16	1.29	4.28%	3.50	0.87061414		Actual Price	153.71
17	2026-11-16	1.29	4.24%	3.75	0.8474939162			

998	2272-02-16	1,107,544,873.05	3.92%	249.16	0.002059131156
999	2272-05-16	1,204,565,803.93	3.92%	249.41	0.002180441421
1000	2272-08-16	1,204,565,803.93	3.92%	249.66	0.002121669169