Basics of Representation Theory

1 Basics

Definition 1. • *Let F be a field. A noncommutative ring A is said to be a finite dimensional F-algebra if $\dim_F A < \infty$ and it is equipped with a ring homomorphism $F \to Z(A)$ taking 1 to 1.*

- *By a finite dimensional module M over A we mean a finite dimensional left A-module.*
- *The algebra A is said to be simple if its only two-sided ideals are 0 and A.*
- *A is said to be a division algebra if $A - \{0\}$ is a group under multiplication in the algebra.*
- *A finite dimensional A-module M is simple if its only A-submodules are 0 and M.*

Definition 2. *If $(A, +, \times)$ is an F-algebra define the opposite F-algebra $(A, +, \times^{\mathrm{op}})$ where the set is A, the addition $+$ is the same as in A but multiplication is $a \times^{\mathrm{op}} b = b \times a$.*

Lemma 3 (Schur)**.** *1. If M and N are simple A-modules and $f \in \mathrm{Hom}_A(M, N)$ then either $f = 0$ or f is an isomorphism.*

2. If M is simple then $\mathrm{End}_A(M)$ is a division algebra.

Proof. Note that $\ker f \subset M$ and $\mathrm{Im}\, f \subset N$ so either $\ker f = 0$ or $\ker f = M$ and either $\mathrm{Im}\, f = 0$ or $\mathrm{Im}\, f = N$. □

Lemma 4. *Let M be a finite-dimensional A-module. The following are equivalent:*

1. *$M = N_1 \oplus \cdots N_r$ where N_i are simple.*
2. *$M = \sum N_i$ with simple $N_i \subset M$.*
3. *If $N \subset M$ then there exists $P \subset M$ such that $M = N \oplus P$.*
4. *If $N \subset M' \subset M$ there exists $P \subset M'$ such that $M' = N \oplus P$.*

Proof. 1 implies 2 is vacuous.

2 implies 3: choose a maximal set of simple submodules $Q_1, \dots, Q_r \subset M$ such that $N + Q_1 + \cdots + Q_r = N \oplus Q_1 \oplus \cdots \oplus Q_r$. If $N \oplus Q_1 \oplus \cdots \oplus Q_r \neq M$ choose a simple $Q_{r+1} \subset M$ such that $Q_{r+1} \not\subset N \oplus Q_1 \oplus \cdots \oplus Q_r$. Since Q_{r+1} is simple it follows that $N \oplus Q_1 \oplus \cdots \oplus Q_r \cap Q_{r+1} = 0$ so $N + Q_1 + \cdots + Q_{r+1} = N \oplus Q_1 \oplus \cdots \oplus Q_{r+1}$ contradicting the maximality of r.

3 implies 4: If $M = N \oplus Q$ then $M' = N \oplus (Q \cap M')$.

4 implies 1: choose $N \subset M$ a simple submodule. Then $M = N \oplus P$ and inductively we get the required decomposition. □

Definition 5. *When the equivalent conditions of the previous lemma hold the module M is said to be semisimple.*

Note 6. *It is left as an exercise that a semisimple module decomposes uniquely (up to reordering) as a direct sum of simple submodules.*

Corollary 7. *Semisimplicity is preserved under direct sums and passage to quotients and submodules.*

Corollary 8. *If A is a semisimple A-module then all finite dimensional A-modules are semisimple. In that case A is said to be a (left) semisimple ring.*

Proof. If M is any finite dimensional A-module then $A^r \twoheadrightarrow M$ for some r and M must be semisimple. $\square$

Corollary 9. *Let M be a finite dimensional A module such that the action of A on M is faithful, i.e., if for $a \in A$ we have $am = 0$ for all $m \in M$ then $a = 0$. If M is semisimple then A is semisimple.*

Corollary 10. *if A is simple as a ring, i.e., there are no nontrivial two-sided ideals, then it is left semisimple.*

Proof. Let $M \subset A$ a simple left A-submodule. Then $\sum_{a \in A} Ma \subset A$ is a two-sided ideal so $\sum_{a \in A} Ma = A$. Since A is then a semisimple A-module it follows that A is a semisimple ring. $\square$

2 Structure of Algebras

Lemma 11 (Wedderburn)**.** *If A is a finite dimensional semisimple F-algebra then*

$$A \cong M_{n_1 \times n_1}(D_1) \oplus \cdots \oplus M_{n_r \times n_r}(D_r)$$

where D_i are division algebras over F.

Any algebra of the above form is semisimple and the expression is unique up to reordering. Moreover, the semisimple modules over A are $D_i^{n_i}$ where the action is given by matrix multiplication.

Proof. Write $A = N_1^{n_1} \oplus \cdots \oplus N_r^{n_r}$ with N_i pairwise nonisomorphic simple modules. Then $\mathrm{End}_A(A) = \oplus \mathrm{End}_A(N_i^{n_i}) = \oplus M_{n_i \times n_i}(D_i)$ where $D_i = \mathrm{End}_A(N_i)$ is a division algebra. Have a natural map $A^{\mathrm{op}} \cong \mathrm{End}_A(A)$ given by $a \mapsto (b \mapsto ba)$ and so

$$A \cong \oplus M_{n_i \times n_i}(D_i^{\mathrm{op}})$$

$\square$

Remark 12. *The above lemma shows that A is simple if and only if $A = M_n(D)$ where D is a division algebra.*

Corollary 13. *If A is a semisimple F-algebra and M and N are finite dimensional A modules then $M \cong N$ if and only if $\mathrm{Tr}\, a|_{\wedge^i M} = \mathrm{Tr}\, a|_{\wedge^i N}$ for all $i \geq 0$ and $a \in A$. If F has characteristic 0 then it is enough to check $\mathrm{Tr}\, a|_M = \mathrm{Tr}\, a|_N$ for all a.*

Proof. Let $M \cong \oplus P_i^{s_i}$ and $N \cong \oplus P_i^{t_i}$ where the P_i are nonisomorphic simple A-modules. Clearly $M \cong N$ if and only if $s_i = t_i$ for all i, if and only if $\dim e_i M = \dim e_i N$ for $i = 1, \ldots, r$ where e_i is the projector onto P_i: $e_i^2 = e_i$, $e_i = 1$ on P_i and $e_i = 0$ on $P_j \neq P_i$. Then $\mathrm{Tr}\, e_i|_{\wedge^j M} = \binom{\dim e_i M}{j}$ for all j and the condition on traces becomes $\binom{s_i \dim P_i}{j} = \binom{t_i \dim P_i}{j}$ for all j.

If F has characteristic 0 then $\mathrm{Tr}\, e_i|_M = s_i \dim P_i$ and $\mathrm{Tr}\, e_i|_N = t_i \dim P_i$ so if $\mathrm{Tr}\, e_i|_M = \mathrm{Tr}\, e_i|_N$ then $s_i = t_i$ for all i. If F has positive characteristic then the condition on traces implies that $(1+x)^{s_i \dim P_i} = (1+x)^{t_i \dim P_i}$ which implies that $s_i = t_i$ for a variable x. $\square$

Definition 14. *The F-algebra A is said to be a central simple algebra if it is a simple finite dimensional algebra such that $F \cong Z(A)$.*

Lemma 15 (Jacobson density theorem)**.** *Let A be a finite dimensional F-algebra and let M be a simple A-module. Let $D = \mathrm{End}_A(M)$ (a division algebra by Schur's lemma). Let $m_1, \ldots, m_r \in M$ be linearly independent over D and let $n_1, \ldots, n_r \in M$. Then there exists $a \in A$ such that $am_i = n_i$ for all i. (In other words, "A is close to $\mathrm{End}_D(M)$".)*

Proof. Let $M = Dm_1 \oplus Dm_2 \oplus \cdots \oplus Dm_r \oplus N$ over D. (This can be done because D is a division algebra, and so it is simple and so M is semisimple as a D-algebra.) Therefore there exists $f \in \mathrm{End}_D(M)$ such that $f(m_i) = n_i$ by linear independence of m_i.

Over A we have $M^r = A(m_1, \dots, m_r) \oplus P$ and $\mathrm{End}_A(M^r) = M_{r\times r}(D)$ so there exists $h \in M_{r\times r}(D)$ which is projection to $A(m_1, \dots, m_r)$. Then

$$\begin{aligned} f \oplus \cdots \oplus f(m_1, \dots, m_r) &= (n_1, \dots, n_r) \\ f \oplus \cdots \oplus f(h(m_1, \dots, m_r)) &= h(f \oplus \cdots \oplus))(m_1, \dots, m_r) \\ &= h(n_1, \dots, n_r) \end{aligned}$$

so $h(n_1, \dots, n_r) \in A(m_1, \dots, m_r)$ and the conclusion follows. □

Lemma 16. *Let A be a central simple K-algebra. Then $A \otimes_K A^{\mathrm{op}} \cong \mathrm{End}_K(A) \cong M_{n\times n}(K)$ where $n = \dim_K A$.*

Proof. $A \otimes_K A^{\mathrm{op}}$ acts on A with a left $\otimes$ right action so get $A \otimes_K A^{\mathrm{op}} \to \mathrm{End}_K(A)$. Let $f \in \mathrm{End}_K A$ and let $a_1, \dots, a_n$ be a basis of A as a K-vector space. Apply the Jacobson density theorem to the $A \otimes_K A^{\mathrm{op}}$-module A. We may do this because A is a simple $A \otimes_K A^{\mathrm{op}}$-module. We get that there exists $c \in A \otimes_K A^{\mathrm{op}}$ such that $ca_i = f(a_i)$ for all i. Therefore c maps to f so $A \otimes_K A^{\mathrm{op}} \twoheadrightarrow \mathrm{End}_K(A)$. A dimension comparison shows that this linear map is an isomorphism. □

Corollary 17. *If A is a central simple K-algebra and B is any simple K-algebra then $A \otimes_K B$ is a simple K-algebra.*

Proof. Let $a_1, \dots, a_n$ be a basis of A/K. For $i = 1, \dots, n$ find $c_i \in A \otimes_K A^{\mathrm{op}}$ with $c_i(a_j) = \delta_{ij}$. Let I be a two-sided ideal of $A \otimes_K B$. If $\sum a_j \otimes b_j \in I$ then $\sum c_i(a_j) \otimes b_j \in I$ so $1 \otimes b_i \in I \cap K \otimes_K B$, where $I \cap K \otimes B$ is a two-sided ideal of B. Since B is simple, either $I \cap K \otimes B = 0$, in which case $b_i = 0$ so $I = 0$, or $I \cap K \otimes B = K \otimes B$ in which case $1 \in I$ so $I = A \otimes B$. □

Corollary 18. *Let A and B be central simple K-algebras. Then $A \otimes_K B$ is also central simple.*

Proof. That $A \otimes B$ is simple follows from the previous corollary. Let a_i be a basis of A/K and let $\sum a_i \otimes b_i \in Z(A \otimes_K B)$. For any $b \in B$ we have $(1 \otimes b)(\sum a_i \otimes b_i) - (\sum a_i \otimes b_i)(1 \otimes b) = \sum a_i \otimes (bb_i - b_i b) = 0$. Therefore $bb_i = b_i b$ for all b so $b_i \in Z(B) = K$. Thus $\sum a_i \otimes b_i \in Z(A \otimes_K K) = Z(A) = K$. □

3 The Brauer Group

Definition 19. *Two central simple K-algebras A and B are equivalent if there exists a division algebra D and two nonnegative integers r and s such that $A \cong M_{r\times r}(D)$ and $B \cong M_{s\times s}(D)$. Let $\mathrm{Br}(K)$ be the set of central simple K-algebras up to equivalence.*

Lemma 20. *The set $\mathrm{Br}(K)$ becomes an abelian group under $\otimes_K$.*

Proof. The identity element is $[K]$ and the inverse of A is A^{op}: $[A][A^{\mathrm{op}}] = [A \otimes_K A^{\mathrm{op}}] = [M_{n\times n}(K)] = [K]$. □

Definition 21. *For L/K a field extension there is a natural map $\mathrm{Br}(K) \to \mathrm{Br}(L)$ given by $[A] \mapsto [A \otimes_K L]$. Let $\mathrm{Br}(L/K) = \ker(\mathrm{Br}(K) \to \mathrm{Br}(L))$.*

Lemma 22 (Double centralizer theorem)**.** *Let A be a central simple K-algebra and let $B \subset A$ be a K-subalgebra. Let $C_A(B) = \{c \in A | cb = bc, \forall b \in B\}$ be the centralizer of B in A. Then1*

1. *$C_A(B)$ is simple.*
2. $\dim_K C_A(B) \dim_K B = \dim_K A$.

3. $C_A(C_A(B)) = B$.

Proof. Since $B \subset A$ it follows there exists n and a division algebra D such that $B \otimes_K A^{\text{op}} = M_{n\times n}(D)$ ($[B][A^{\text{op}}] = [K]$). Therefore there exists an integer r such that $A \cong (D^n)^r$ as a $M_{n\times n}(D) = B \otimes_K A^{\text{op}}$ module. Note that $C_A(B) = \text{End}_{B\otimes_K A^{\text{op}}}(A)$ ($A \cong \text{End}_{A\otimes_K A^{\text{op}}}(A)$). But $\text{End}_{B\otimes_K A^{\text{op}}}(A) = M_{r\times r}(D^{\text{op}})$ which implies that $C_A(B)$ is simple, as matrix algebras are simple.

Also, $\dim_K C_A(B) = r^2 \dim_K D^{\text{op}} = r^2 \dim_K D$ and $\dim_K B \dim_K A = \dim_K(B \otimes_K A^{\text{op}}) = n^2 \dim_K D$. Therefore $\dim_K A = rn \dim_K D$ which implies the second part.

Finally, $B \subset C_A(C_A(B))$ and a dimension comparison implies isomorphism. □

Corollary 23. *Let D/K be a division algebra. Then $\dim_K D$ is a square number and any maximal subfield of D has dimension $\sqrt{\dim_K D}$.*

Proof. Let $L \subset D$ be a maximal subfield. Then $C_D(L) \subset L$. If $L \neq C_D(L)$ choose $x \in C_D(L) - L$ in which case $L(x)$ is a commutative division algebra, so a field, which contradicts the choice of L. Therefore $L = C_D(L)$ and the previous lemma implies that $(\dim_K L)^2 = \dim_K D$. □

Corollary 24. *Let A be a central simple K-algebra and let L be a maximal subfield of A. Then $A \otimes_K L \cong M_{n\times n}(L)$ for some n, i.e., $[A] \in \text{Br}(L/K)$.*

Proof. Let $L \subset C_A(L) \cong M_{r\times r}(D)$ for some division algebra D. Then $L \subset Z(C_A(L)) = D$ so $L \subset D$. Again, by maximality of L we deduce that $L = D$ so $C_A(L) \cong M_{r\times r}(L)$, but this implies (as in the proof of the double centralizer theorem) that $L \otimes_K A^{\text{op}} \cong M_{n\times n}(L)$. □