Basics of Homological Algebra

1 Definitions

Definition 1. *A category A is said to be abelian if for any objects M and N,* $\operatorname{Hom}(M,N)$ *is an abelian group, kernels and cokernels exist in the category and if finite direct sums exist in the category.*

Example 2. *The categories of sets, the category* Ab-Gr *of abelian groups and the category* Mod_R *of modules over a fixed ring R are all abelian categories.*

Definition 3. *Let A be an abelian category. A sequence $K^\bullet = \cdots \to M_{i-1} \overset{d_{i-1}}{\to} M_i \overset{d_i}{\to} M_{i+1} \to \cdots$ is said to be a complex if $d_i \circ d_{i-1} = 0$ for all i. The complex $K^\bullet$ is said to be exact at M_i if $\operatorname{Im} d_{i-1} = \ker d_i$; the complex is said to be exact if it is exact at all i. The cohomology of $K^\bullet$ is $H^i(K^\bullet) = \ker d_i / \operatorname{Im} d_{i-1}$; the cohomology groups fit into a complex $H^\bullet(K^\bullet) = \cdots \to H^{i-1}(K^\bullet) \to H^i(K^\bullet) \to \cdots$.*

Remark 4. *The complex $0 \to M \to N$ is exact at M if the map $M \to N$ is injective. The complex $M \to N \to 0$ is exact at N if $M \to N$ is surjective.*

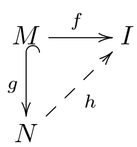
Definition 5. *A covariant functor $F : A \to B$ between two abelian categories is said to be left-exact if for all exact sequences $0 \to M \to N \to P \to 0$ in A one gets an exact sequence $0 \to F(M) \to F(N) \to F(P)$ in B.*

Example 6. *Let G be a profinite group and let $\mathbb{Z}[G]$ be the ring generated linearly over $\mathbb{Z}$ by elements of G, with multiplication extended by linearity from multiplication in the group G. Let* Mod_G *be the category of continuous $\mathbb{Z}[G]$-modules, where a $\mathbb{Z}[G]$-module M is said to be continuous if for all $m \in M$ the stabilizer* $\mathrm{Stab}_G(m) = \{g \in G | gm = m\}$ *is an open subgroup of G. Then* Mod_G *is an abelian category and the functor* $F : \mathrm{Mod}_G \to$ Ab-Gr *given by $F(M) = M^G = \{m \in M | gm = m, \forall g \in G\}$ is left-exact.*

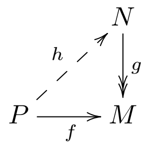
Definition 7. *Let A and B be two categories and let $F, G : A \to B$ be two functors. A natural transformation $f : F \to G$ is the datum, for each object M in A of a morphism $f_M : F(M) \to G(M)$ such that if $M \to N$ is a morphism then we get a commutative diagram*

$$\begin{array}{ccc} F(M) & \xrightarrow{f_M} & G(M) \\ \downarrow & & \downarrow \\ F(N) & \xrightarrow{f_N} & G(N) \end{array}$$

Definition 8. *Let A be an abelian category. An object I of A is said to be injective if for all morphisms f and all injective morphisms g there exists a morphism h making the diagram commutative*



An object P of A is said to be projective if for all morphisms f and all surjective morphisms g there exists a morphism h making the diagram commutative



A category A is said to have "enough injectives" if for every object M of A there exists an injective object I of A and an injection $A \hookrightarrow I$. The category A has "enough projectives" if for every object M of A there exists a projective object P of A and a surjection $P \twoheadrightarrow M$.

Example 9. *In the category $\mathrm{Mod}_{\mathbb{Z}}$ of abelian groups (or equivalently of modules over $\mathbb{Z}$) the object $\mathbb{Z}$ is not injective because while $\mathbb{Z}$ injects into $\mathbb{Q}$ there is no morphism $\mathbb{Q} \to \mathbb{Z}$. However, the object $\mathbb{Q}/\mathbb{Z}$ is injective.*

Lemma 10. *Let A be a category with enough injectives. Then every object M of A has an injective resolution, i.e., there exists an exact sequence*

$$0 \to M \to I_0 \to I_1 \to \dots$$

where I_n are injectives. We write such a resolution as $0 \to M \to I^\bullet$.

Proof. Assume we inductively have an exact sequence

$$0 \to M \to I_0 \to \dots \to I_k$$

where I_k are injective. For $k = 1$ this follows from the fact that the category has enough injectives. Consider an injection $I_k / \operatorname{Im} I_{k-1}$ into some injective object I_{k+1}. Then

$$0 \to M \to I_0 \to \dots \to I_k \to I_{k+1}$$

will be exact. In the limit we get the desired injective resolution. □

Lemma 11. *Given two injective resolutions $0 \to M \to I_1^\bullet$ and $0 \to M \to I_2^\bullet$ we get an isomorphism of complexes $H^\bullet(I_1^\bullet) \cong H^\bullet(I_2^\bullet)$, in other words, for each $i \geq 0$ we have $H^0(I_1^\bullet) \cong H^0(I_2^\bullet)$.*

Proof. One can show that there exist maps of complexes $\alpha : I_1^\bullet \to I_2^\bullet$ and $\beta : I_2^\bullet \to I_1^\bullet$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to the identity. Recall that a map of complexes $f : K^\bullet \to K^\bullet$ is homotopic to 0 (f and g are homotopic if $f - g$ is homotopic to 0) if it induces the 0 map on cohomology. For details, see Eisenbud *Commutative algebra with a view towards algebraic geometry* Appendix A.3. □

Proposition 12. *Let $F : A \to B$ be a covariant left-exact functor between two abelian categories with A having enough injectives. Then there exist covariant functors $R^i F : A \to B$ called the right derived functors of F such that for all exact sequences $0 \to M \to N \to P \to 0$ we get an exact sequence*

$$0 \to F(M) \to F(N) \to F(P) \to R^1F(M) \to R^1F(N) \to R^1F(P) \to R^2F(M) \to \dots$$

Proof. Let M be an object of A and let $0 \to M \to I^\bullet$ be an injective resolution. Define $R^iF(M) = H^i(F(I^\bullet))$ where H^i is the i-th cohomology group of the complex $I^\bullet$. The previous lemma implies that this definition is independent of the choice of injective resolution $I^\bullet$. □

Example 13. *Let G be a profinite group and let $F : \mathrm{Mod}_G \to \text{Ab-Gr}$ given by $F(M) = M^H$. The group cohomology $H^i(G, -)$ is the i-th right-derived functor of F.*

Definition 14. *Let $F : A \to B$ be a left-exact functor between two abelian categories. An object M of A is said to be F-acyclic if $R^iF(M) = 0$ for $i > 0$. In particular, if G is a profinite group an object M of Mod_G is acyclic if $H^i(G, M) = 0$ for $i > 0$.*

Note 15. *If I is an injective module in Mod_G then I is acyclic.*

Proof. An injective resolution for I is simply $0 \to I \to I \to 0$ and so $0 \to I^G \to I^G \to 0$ is still exact. Thus $H^i(G, I) = 0$ for $i > 0$. $\square$

Lemma 16. *Let M be a module in Mod_G and let $0 \to M \to J^\bullet$ be a resolution such that J_n is acyclic for all n. Then $H^i(G, M) = H^i(J^\bullet)$.*

Proof. This is easy. $\square$

Definition 17. *A covariant functor $F : A \to B$ between two abelian categories A and B is said to be effaceable if for any object M of A there exists an injection $M \hookrightarrow N$ in A such that $F(M) \to F(N)$ is the 0 morphism.*

Example 18. *If G is a profinite group and $F(M) = M^G$ as before the $H^i(G, -) = R^iF$ is effaceable for $i > 0$.*

Proof. Let I be an injective object containing M. It is injective so acyclic and therefore $H^i(G, M) \to H^i(G, I) = 0$. $\square$

2 Covariant δ-functors

Definition 19. *By a covariant δ-functor between two abelian categories A and B we mean a series of covariant functors $H^i : A \to B$ with the property that for each exact sequence $0 \to M \to N \to P \to 0$ in A we get homomorphisms $\delta_i : H^i(P) \to H^{i+1}(M)$ such that*

1. *If $0 \to M \to N \to P \to 0$ is exact then get an exact sequence*

$$0 \to H^0(M) \to H^0(N) \to H^0(P) \xrightarrow{\delta_0} H^1(M) \to H^1(N) \to H^1(P) \xrightarrow{\delta_1} H^2(M) \to \cdots$$

2. *If*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & P' & \longrightarrow & 0 \end{array}$$

are two exact sequences such that the diagram commutes then we get a commutative diagram

$$\begin{array}{ccc} H^i(P) & \xrightarrow{\delta_i} & H^{i+1}(M) \\ \downarrow & & \downarrow \\ H^i(P') & \xrightarrow{\delta_i} & H^{i+1}(M') \end{array}$$

A δ-functor is called universal if for any other δ-functor $(\overline{H}^i, \overline{\delta}_i)$ and any natural transformation $f : H^0 \to \overline{H}^0$ there exist unique natural transformations $f^i : H^i \to \overline{H}^i$ commuting with the δ_i-s.

Example 20. *Let F be a left-exact covariant functor. Then the right derived functors R^iF form a δ-functor.*

Lemma 21. *Let $H = (H^i, \delta_i)$ be a δ-functor such that H^i is effaceable for $i > 0$. Then H is a universal δ-functor.*

Corollary 22. *If G is a profinite group then $H^i(G, -), \delta_i)$ is a universal δ-functor.*

3 Spectral sequences

Definition 23. *A spectral sequence is a collection of objects $E_i^{p,q}$ of an abelian category for $i \geq 1$ and $p, q \in \mathbb{Z}$, together with morphisms $d_i^{p,q} : E_i^{p,q} \to E_i^{p+i,q-i+1}$ such that $d_i \circ d_i = 0$, such that $E_{i+1}^{p,q}$ is the cohomology group of the complex $\ldots \to E_i^{p-i,q+i-1} \to E_i^{p,q} \to E_i^{p+i,q-i+1} \to \ldots$ at the position of $E_i^{p,q}$.*

Definition 24. *A spectral sequence $E_i^{p,q}$ stabilizes if for large i we have $E_i^{p,q} = E_\infty^{p,q}$ is independent of i.*

Definition 25. *A spectral sequence $E_i^{p,q}$ abuts to E^{p+q} in which case we write $E_i^{p,q} \Longrightarrow E^{p+q}$ if it stabilizes and if there exists a filtration $E^{p+q} = \mathrm{Fil}^0 \supset \mathrm{Fil}^1 \supset \ldots \supset \mathrm{Fil}^k = 0$ such that $E_\infty^{i,p+q-i} \cong \mathrm{Fil}^i / \mathrm{Fil}^{i+1}$.*

Theorem 26 (Grothendieck spectral sequence)**.** *Let A, B and C be three abelian categories and let $F : A \to B$ and $G : B \to C$ be covariant left-exact functors such that if I is an injective object of A then $F(I)$ is G-acyclic. Then there exists a spectral sequence $E_2^{p,q} = R^pG \circ R^qF \Longrightarrow R^{p+q}(G \circ F)$.*