Homework 1

Due Friday, January 13, 2012

Homework is due at 4 PM on the following Friday. Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

In the following K is a field of characteristic 0 and v is a discrete valuation on K.

1. Assume $x = \dfrac{1+\sqrt{-3}}{2} \in K$. What is $v(x)$?
2. Show that $\sqrt{7} \in \mathbb{Q}_3$.
3. Show that $\sqrt{17} \in \mathbb{Q}_2$. [Hint: use the binomial expansion of $(1+16)^{1/2}$.]
4. Let $p > 2$ be a prime number and let $A \in \mathrm{GL}(n, \mathbb{Z}_p)$ a matrix of finite order such that $A \equiv I_n \pmod{p}$. Show that $A = I_n$. [Hint: for A of prime order q look at $(I_n + pX)^q$, treating $q = p$ and $q \neq p$ separately.]
5. Show that if $f \in K[X]$ is a polynomial such that NP_f is pure and contains no points with integer coordinates other than the endpoints then f is irreducible. Deduce the Eisenstein irreducibility criterion.
6. Which of the following polynomials are irreducible over $\mathbb{Q}_5$:
   (a) $x^2 + 1$;
   (b) $x^2 + 2$;
   (c) $x^2 + 5$;
   (d) $x^3 + 5x + 25$;
   (e) $x^3 + 5x^2 + 25$;
   (f) $x^4 + 5x^2 + 25$;
   (g) $x^4 + 15x^2 + 25$?

   [Note: not all of these are amenable to Newton polygons.]