Homework 2

Due Tuesday, January 24

Because I missed class last Friday and Monday was off, this homework is due in on Tuesday, January 24, at 4 PM (I will have office hours the day before).

Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

For problems 3, 4, 5 and 6 please do not consult Serre's Local Fields.

1. Let L/K be a finite extension of local fields, let ϖ_L be a uniformizer for L and let $\overline{\beta}$ such that $k_L = k_K(\overline{\beta})$. Let $\omega : k_L^\times \to \mathcal{O}_L^\times$ be the Teichmüller lift (with the convention that $\omega(0) = 0$). Show that

$$\{\varpi_L^i \omega(\overline{\beta})^j\}_{0 \le i < e_{L/K}, 0 \le j < f_{L/K}}$$

are linearly independent over K.

2. (Krasner's Lemma) Let K be a field of characteristic 0 which is complete with respect to the valuation v.
   (a) Let L/K be a finite Galois extension and $\alpha, \beta \in L$. If

   $$v(\beta - \sigma\alpha) < v(\beta - \alpha)$$

   for all $\sigma \in \mathrm{Gal}(L/K) - \mathrm{Gal}(L/K(\alpha))$ show that $\alpha \in K(\beta)$.
   (b) If $f \in K[x]$ has degree d show that there exist real numbers a, b such that if $g \in K[x]$ has degree d and $v_0(f - g) > a$ then any root β of g satisfies $v(\beta) > b$.
   (c) If $f \in K[x]$ is irreducible of degree d show that there exists a constant c such that if $g \in K[x]$ has degree d and $v_0(f - g) > c$ then g is irreducible and $K[x]/(g) \cong K[x]/(f)$.
   (d) Find the Galois group of the splitting field of $x^4 - 10x^2 + 27x + 1$ over $\mathbb{Q}_3$. [Hint: $\sqrt{2} + \sqrt{3}$ is a root of $x^4 - 10x^2 + 1$.]

3. Let $K = \mathbb{Q}_2[\sqrt{-1}, \sqrt{-2}]$.
   (a) Find a uniformizer for K.
   (b) Compute $\mathrm{Gal}(K/\mathbb{Q}_2)$.

4. Let p be a prime number.
   (a) Show that $f(x) = x^p - x - 1/p$ is irreducible over $\mathbb{Q}_p$.
   (b) Let α be a root. For $0 \le i < p$ show that there exists a unique $\beta_i \in \mathcal{O}_{\mathbb{Q}_p(\alpha)}$ such that $\alpha + \beta_i$ is a root of $f(x)$ and $\beta_i \equiv i \pmod{\wp_{\mathbb{Q}_p(\alpha)}}$. [Hint: Hensel's Lemma. But beware that Hensel's lemma only applies to monic polynomials with integral coefficients.]
   (c) Deduce that $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is Galois.

5. Let $p > 2$ be a prime number and let ζ be a primitive p^n-th root of unity. Let $K = \mathbb{Q}_p(\zeta)$.
   (a) Find a uniformizer for K.

(b) Show that $K/\mathbb{Q}_p$ is Galois and find the Galois group.

6. Let $p > 2$ be a prime number. Let $K/\mathbb{Q}_p$ be a totally ramified Galois extension of degree p. Let v_K be a valuation on K such that $\operatorname{Im} v_K = \mathbb{Z}$ (necessarily then $v_K \nmid v_p!$). Let ϖ_K be a uniformizer and let $f(x) = x^p + a_{p-1}x^{p-1} + \cdots + a_0 \in \mathbb{Z}_p[x]$ be its minimal polynomial. Show that $v_K(\mathcal{D}_{K/\mathbb{Q}_p}) = v_K(f'(\varpi_K))$ is equal to the minimum of $2p-1$ and $v_K(a_i) + i - 1$ for $1 \leq i < p$.