Homework 3

Due Tuesday, January 31

In this problem set you will finish some of the exercises from the previous one.

Homework is due the following Tuesday, at 4 PM. Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

For these exercises please do not consult Serre's Local Fields.

1. Let $K = \mathbb{Q}_2[\sqrt{-1}, \sqrt{-2}]$.
   (a) For $u \geq -1$ find $\mathrm{Gal}(K/\mathbb{Q}_2)_u$ and $\mathrm{Gal}(K/\mathbb{Q}_2)^u$.
   (b) Find $\mathcal{D}_{K/\mathbb{Q}_2}$. [Hint: don't compute derivatives.]
2. Let p be a prime number, let $f(x) = x^p - x - 1/p$ and let α be a root of f. Recall from the previous homework that f is irreducible over $\mathbb{Q}_p$ and that $\mathbb{Q}_p(\alpha)/\mathbb{Q}_p$ is Galois. Compute $\mathrm{Gal}(\mathbb{Q}_p(\alpha)/\mathbb{Q}_p)_u$ for all $u \geq -1$.
3. Let $p > 2$ be a prime number and let ζ be a primitive p^n-th root of unity. Let $K = \mathbb{Q}_p(\zeta)$.
   (a) Determine $e_{K/\mathbb{Q}_p}$ and $f_{K/\mathbb{Q}_p}$.
   (b) Determine $\mathrm{Gal}(K/\mathbb{Q}_p)_u$ and $\mathrm{Gal}(K/\mathbb{Q}_p)^u$ for all $u \geq -1$.
4. Let $p > 2$ be a prime number. Let $K/\mathbb{Q}_p$ be a totally ramified Galois extension of degree p. Let v_K be a valuation on K such that $\mathrm{Im}\, v_K = \mathbb{Z}$. Let ϖ_K be a uniformizer and let $f(x) = x^p + a_{p-1}x^{p-1} + \cdots + a_0 \in \mathbb{Z}_p[x]$ be its minimal polynomial. In the previous homework you showed that $v_K(\mathcal{D}_{K/\mathbb{Q}_p})$ is equal to the minimum of $2p - 1$ and $v_K(a_i) + i - 1$ for $1 \leq i < p$.
   (a) Show that $v_K(\mathcal{D}_{K/\mathbb{Q}_p}) \equiv 0 \pmod{p-1}$.
   (b) Deduce that $v_K(\mathcal{D}_{K/\mathbb{Q}_p}) = 2p - 2$.
   (c) Compute $\mathrm{Gal}(K/\mathbb{Q}_p)_u$ for $u \geq -1$.
   (d) Compute $\mathrm{Gal}(K/\mathbb{Q}_p)^u$ for $u \geq -1$.
   (e) Suppose $L/\mathbb{Q}_p$ is totally ramified and Galois with Galois group $(\mathbb{Z}/p\mathbb{Z})^2$. Show that $L = K_1K_2$ where $K_i/\mathbb{Q}_p$ is totally ramified Galois of degree p. Show that
   $$\mathrm{Gal}(L/\mathbb{Q}_p)^u \hookrightarrow \mathrm{Gal}(K_1/\mathbb{Q}_p)^u \times \mathrm{Gal}(K_2/\mathbb{Q}_p)^u$$
   for all $u \geq -1$. Compute $\mathrm{Gal}(L/\mathbb{Q}_p)^u$ for all $u \geq -1$ and $\mathrm{Gal}(L/\mathbb{Q}_p)_1/\mathrm{Gal}(L/\mathbb{Q}_p)_2$. Derive a contradiction and conclude that no such L exists.
5. Let $p > 2$ be a prime.
   (a) Show that there exist three isomorphism classes of quadratic extensions of $\mathbb{Q}_p$:
      - the unique unramified extension of $\mathbb{Q}_p$ of degree 2, explicitly $\mathbb{Q}_p(\sqrt{u})$ for some $u \in \mathbb{Z}_p^\times$ such that the image of u in $\mathbb{F}_p^\times$ is not a square;
      - the ramified extension $F_1 = \mathbb{Q}_p(\sqrt{p})$;

- and the ramified extension $F_2 = \mathbb{Q}_p(\sqrt{up})$ where $u \in \mathbb{Z}_p^\times$ such that the image of u in $\mathbb{F}_p^\times$ is not a square.

(b) Let $F = F_1 F_2$.

i. Show that $[F : \mathbb{Q}_p] = 4$.

ii. Show that $2 \mid f_{F/\mathbb{Q}_p}$ by looking at residue fields.

iii. Deduce that $e_{F/\mathbb{Q}_p} = f_{F/\mathbb{Q}_p} = 2$.

iv. Show that $f_{F/F_i} = 1$ for $i = 1, 2$ which shows that the composite of two totally ramified fields need not be totally ramified over the two fields which define the composite.