Homework 4

Due Tuesday, February 7

Homework is due the following Tuesday, at 4 PM. Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

In the following K will be a field and all K-algebras will be finite dimensional K-algebras. You may use the Noether-Skolem theorem which says that if A is a central simple K-algebra, B is a simple K-algebra and j_1, j_2 are embeddings $B \hookrightarrow A$ of K-algebras then j_1 and j_2 are conjugate by an element of $K^\times$.

1. Let $L/K/\mathbb{Q}_p$ be finite extensions. Show that

$$v_p(\mathcal{D}_{L/K}) = \frac{1}{e_{K/\mathbb{Q}_p}} \int_{-1}^{\infty} \left(1 - \frac{1}{\# G^u_{L/K}}\right) du$$

   [Note: this is actually useful.]

2. (a) If K is an algebraically closed field show that the Brauer group of K is trivial.
   (b) Show that the Brauer group of $\mathbb{R}$ is $\mathbb{Z}/2\mathbb{Z}$. [Hint: If D is a central division algebra over $\mathbb{R}$ choose a maximal subfield L and an element $j \in D - L$ which normalises L and determine the relations they need to satisfy.]
   (c) Show that the Brauer group of a finite field K is trivial. [Hint: If D is a central division algebra, choose a maximal subfield L and show that $D^\times = \bigcup_{g \in D^\times/L^\times} gL^\times g^{-1}$.]

3. Let A be a central simple K-algebra of dimension n^2 and assume that K is perfect (i.e., that every algebraic extension is separable).
   (a) Show that there exists an isomorphism $\phi : A \otimes_K \overline{K} \cong M_{n\times n}(\overline{K})$.
   (b) If $\sigma \in \mathrm{Gal}(\overline{K}/K)$ show that $\sigma \circ \phi|_A$ is $\mathrm{GL}(n, \overline{K})$-conjugate to $\phi|_A$.
   (c) Deduce that for $a \in A$ the characteristic polynomial of $\phi(a)$ lies in $K[x]$ and show that it does not depend on the choice of ϕ. This polynomial is called the characteristic polynomial of a. Minus the coefficient of x^{n-1} is called the reduced trace $\mathrm{Tr}_{A/K}\, a$ of a and $(-1)^n$ times the constant coefficient is called the reduced norm $N_{A/K} a$ of a.
   (d) Show that if $a, a' \in A$ and $c \in K$ then $\mathrm{Tr}_{A/K}(ca + a') = c\,\mathrm{Tr}_{A/K}\, a + \mathrm{Tr}_{A/K}\, a'$.
   (e) Show that if $a, a' \in A$ then $N_{A/K}(aa') = (N_{A/K} a)(N_{A/K} a')$.
   (f) Show that if $a \in A$ and $K(a)$ is a field then

$$\mathrm{Tr}_{A/K}\, a = \left(\frac{n}{[K(a) : K]}\right) \mathrm{Tr}_{K(a)/K}\, a$$
$$N_{A/K} a = \left(N_{K(a)/K} a\right)^{n/[K(a):K]}$$