Homework 5

Due Tuesday, February 14

Homework is due the following Tuesday, at 4 PM. Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

1. Let K be a local field.
   (a) Let D be a central division algebra over K^{ur}. Show that v_K extends to a homomorphism $D^\times \to \mathbb{Q}$ satisfying $v_K(1+d) \geq \min(0, v_K(d))$ for all $d \in D$.
   (b) Let $\mathcal{O}_D$ be the subset of $d \in D$ with $v_K(d) \geq 0$ and let $\Pi \in \mathcal{O}_D$ be an element with minimal valuation > 0. Show that $\mathcal{O}_D$ is a (noncommutative) ring, that $D = \mathcal{O}_D[1/\varpi_K]$, that $\mathcal{O}_D = \mathcal{O}_{K^{\mathrm{ur}}} + \Pi\mathcal{O}_D$ and hence that $\mathcal{O}_{K^{\mathrm{ur}}}[\Pi]$ is dense in $\mathcal{O}_D$.
   (c) Deduce that $\mathrm{Br}(K^{\mathrm{ur}}) = 0$.
2. (Injective modules)
   (a) Show that in the category of abelian groups (in other words, of $\mathbb{Z}$-modules, or G-modules where $G = \{1\}$) the objects $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective, but $\mathbb{Z}$ is not.
   (b) If $G = \{\pm 1\}$ find an injective G-module which contains $\mathbb{Z}$ with G acting trivially. Also find an injective G-module which contains $\mathbb{Z}/2\mathbb{Z}$ with trivial G-action.
3. (Right derived functors) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ has enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant left-exact functor. For an object M of $\mathcal{A}$ let $0 \to M \to I_0 \to I_1 \to \ldots$ be an injective resolution and let $R^iF(M) = \ker(F(I_i) \to F(I_{i+1}))/\operatorname{Im}(F(I_{i-1}) \to F(I_i))$ (where we write $R^0F(M) = F(M)$).
   (a) Show that $R^iF(M)$ is independent of a choice of injective resolution.
   (b) Show that for any exact sequence $0 \to M \to N \to P \to 0$ we get a (long) exact sequence
   $$0 \to F(M) \to F(N) \to F(P) \xrightarrow{\delta_0} R^1F(M) \to R^1F(N) \to R^1F(P) \xrightarrow{\delta_1} \ldots$$
   $$\ldots \to R^iF(M) \to R^iF(N) \to R^iF(P) \xrightarrow{\delta_i} R^{i+1}F(M) \to \ldots$$
   (c) Show that instead of an injective resolution one can also use an acyclic resolution, i.e., a resolution $0 \to M \to J_0 \to J_1 \to \ldots$ such that for all $i \geq 0$ the object J_i is F-acyclic.
4. Let G be a profinite group and H a closed subgroup. Show that $\mathrm{Ind}_H^G : \mathrm{Mod}_H \to \mathrm{Mod}_G$ is an exact functor.
5. Give an example of a finite group G and an exact sequence of G-modules $0 \to M \to N \to P \to 0$ such that $0 \to M^G \to N^G \to P^G$ is not right exact.