Homework 6

Due Tuesday, February 21

Homework is due the following Tuesday, at 4 PM. Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

1. Let G be a profinite group and let H be an open subgroup. Show that $\mathrm{cor}_H^G \circ \mathrm{res}_H^G : H^i(G,M) \to H^i(G,M)$ is $[G:H]$.

2. Let $G = \widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$ and let $F \in G$ correspond to $1 \in \widehat{\mathbb{Z}}$. Let M be a torsion G-module.

   (a) Show that
   $$\begin{aligned} Z^1(G,M) &\to M \\ \varphi &\mapsto \varphi(F) \end{aligned}$$
   is an isomorphism. Show that $H^1(G,M) \cong M/(F-1)M$.

   (b) Show that for $i > 1$ we have $H^i(G,M) = 0$. [Hint: use dimension shifting.]

3. Let G be a profinite group and let $0 \to M \to N \to P \to 0$ be an exact sequence in Mod_G. For $x \in P$ let $\overline{x} \in N$ be any lift. Show that the connecting homomorphisms $\delta_i : H^i(G,P) \to H^{i+1}(G,M)$ can be expressed as $\delta_i[\phi] = [\psi]$ where
   $$\psi(g_1,\dots,g_{i+1}) = g_1\overline{(\phi(g_2,\dots,g_{i+1}))} + \sum_{j=1}^{i}(-1)^j\overline{(\phi(\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots))} + (-1)^{i+1}\overline{(\phi(g_1,\dots,g_i))}$$

4. Use the previous characterization of δ_i to show that

   (a) if $H \subset G$ is a closed subgroup of a profinite group and $\phi \in Z^i(G,M)$ then $\mathrm{res}_H^G[\phi] = [\phi|_{H^i}]$.

   (b) if $H \subset G$ is a closed normal subgroup of a profinite group and $\phi \in Z^i(G/H, M^H)$ then $\mathrm{inf}_H^G[\phi] = [\phi \circ (G^i \to (G/H)^i)]$.

5. Let G be a profinite group and M a discrete G-module. Suppose $M = M_0 \supset M_1 \supset M_2 \supset \dots$ are G-submodules with $\cap M_i = \{0\}$ and that M is complete with respect to the topology induced by the M_i (i.e., if $m_i \in M_i$ satisfy $m_i - m_j \in M_i$ for all $j \geq i$ then there exists $m \in M$ such that $m - m_i \in M_i$ for all i). If $H^i(G, M_j/M_{j+1}) = 0$ for all $j \geq 0$ show that $H^i(G,M) = 0$.

   [Hint: Let $\phi \in Z^i(G,M)$. Show by induction on r that one can find $\psi_r \in C^{i-1}(G,M)$ with $\psi_r - \psi_{r-1} \in C^{i-1}(G,M)$ and $d\psi_r \equiv \phi \pmod{M_r}$.]