Homework 7

Due Tuesday, February 28, 2012

Homework is due the following Tuesday, at 4 PM. Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

1. Let G be a profinite group and let H be a closed subgroup.

   (a) Show that under the canonical isomorphism $H^i(G, \operatorname{Ind}_H^G M) \cong H^i(H, M)$ the restriction map $H^i(G, M) \to H^i(H, M)$ becomes identified with the map $H^i(G, M) \to H^i(G, \operatorname{Ind}_H^G M)$ induced by

   $$\begin{aligned} M &\to \operatorname{Ind}_H^G M \\ m &\mapsto (g \mapsto gm) \end{aligned}$$

   (b) If H is open show further that the corestriction map $H^i(H, M) \to H^i(G, M)$ becomes identified with the map $H^i(G, \operatorname{Ind}_H^G M) \to H^i(G, M)$ induced by

   $$\begin{aligned} \operatorname{Ind}_H^G M &\to M \\ \phi &\mapsto \sum_{g \in G/H} g^{-1}\phi(g) \end{aligned}$$

2. In this problem you will finish the proof that if $[L : K] = n$ then the connecting homomorphism $\delta : H^1(G_{L/K}, \mathrm{PGL}(n, L)) \to H^2(G_{L/K}, L^\times)$ is surjective (recall that δ arises in the cohomology long exact sequence associated to the exact sequence $1 \to L^\times \to \mathrm{GL}(n, L) \to \mathrm{PGL}(n, L) \to 1$). For $\phi \in H^2(G_{L/K}, L^\times)$ we defined the function $\psi : G_{L/K} \to \mathrm{GL}(V)$ where $V = L[G_{L/K}]$ by

   $$\psi(\sigma)(\sum a_\tau[\tau]) = \sum a_\tau \phi(\sigma, \tau)[\sigma\tau]$$

   (a) Show that $\psi \in H^1(G_{L/K}, \mathrm{PGL}(n, L))$, i.e., that it is a cochain.

   (b) Show that

   $$\phi(\sigma, \tau) = \psi(\sigma)\sigma(\psi(\tau))\psi(\sigma\tau)^{-1}$$

   and use the previous problem set to deduce that $\delta[\psi] = [\phi]$.

3. Let G be a profinite group, H a closed subgroup of G and I a normal closed subgroup of G. Let M be a discrete G/I-module.

   (a) Show that the diagram

   $$\begin{array}{ccc} H^i(G/I, M) & \xrightarrow{\text{inf}} & H^i(G, M) \\ \Big\downarrow{\scriptstyle\text{res}} & & \Big\downarrow{\scriptstyle\text{res}} \\ H^i(H/(I \cap H), M) & \xrightarrow{\text{inf}} & H^i(H, M) \end{array}$$

   is commutative.

(b) Suppose that H is also open in G and I acts trivially on M. Show that the diagram

$$\begin{array}{ccc} H^i(G/I, M) & \xrightarrow{\text{inf}} & H^i(G, M) \\ \uparrow{\scriptstyle \text{cor}} & & \uparrow{\scriptstyle [I:H\cap I]\cdot\text{cor}} \\ H^i(H/(I\cap H), M) & \xrightarrow{\text{inf}} & H^i(H, M) \end{array}$$

is commutative, where the left vertical map is corestriction and the right vertical map is $[H : H \cap I]$ times corestriction.

4. Let $E_2^{i,j} \implies E^n$ be a spectral sequence supported on the first quadrant (i.e., $E_2^{i,j} = 0$ unless $i, j \geq 0$). Show that there exists an exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to \ker\left(E^2 \to E_2^{0,2}\right) \to E_2^{1,1} \to E_2^{3,0}$$

[Hint: Show that $\ker\left(E^2 \to E_2^{0,2}\right) = \ker\left(E^2 \to E_\infty^{0,2}\right)$.]

5. Let G be a pro-p group and M a finite discrete G-module with $p \nmid \#M$. Show that $M^G \cong M_G$ where $M_G = M/N$ where N is the smallest abelian subgroup of M generated by $\langle (g-1)m, g \in G, m \in M \rangle$ are the coinvariants. [Hint: let $H \subset G$ be a finite index subgroup which acts trivially on M then show that the kernel of the natural map $M \to M_G$ is the same as the kernel of the map $M \to M^G$ given by $m \mapsto \sum_{g \in G/H} gm$.]

6. Suppose that G is a profinite group and that H is an open subgroup of G. Suppose also that $t : H\backslash G \to G$ is a section to the natural projection.

(a) Suppose that M is a discrete G-module and that $\phi \in Z^1(H, M)$. Define $\Phi : G \to M$ by

$$\Phi(g) = \sum_{x \in H\backslash G} t(x)^{-1}\phi(t(x)gt(xg)^{-1})$$

Show that $\Phi \in Z^1(G, M)$ and that $[\Phi] = \text{cor}_H^G[\phi]$ in $H^1(G, M)$.

(b) Show that

$$V(g) = \prod_{x \in H\backslash G} t(x)gt(xg)^{-1}$$

defines a homomorphism (the "transfer map") $V : G^{\text{ab}} \to H^{\text{ab}}$ such that

$$\phi(V(g)) = (\text{cor}_H^G \phi)(g)$$

for all $g \in G$ and $\phi \in H^1(H, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(H, \mathbb{Q}/\mathbb{Z})$.