Homework 8

Due Friday, March 9

This homework is due on the last day of class, Friday, March 9, at 4 PM. Since the lowest homework grade will be dropped, no late homework is accepted. You are encouraged to work together with others, but you must write up the solutions on your own. (In particular, you may not simply say that some problem is the content of Proposition X from book Y.)

1. Let $K/\mathbb{Q}_p$ be a finite extension and let M be a finite discrete G_K module. Let
$$\langle \cdot, \cdot \rangle : H^1(G_K, M) \times H^1(G_K, M^*) \to \mathbb{Q}/\mathbb{Z}$$
be the local Tate pairing and let $H^1_{\mathrm{ur}}(K, M) := H^1(G_{K^{\mathrm{ur}}/K}, M^{I_K})$ (sitting inside $H^1(G_K, M)$ via inflation) and $H^1_{\mathrm{ur}}(K, M^*) := H^1(G_{K^{\mathrm{ur}}/K}, (M^*)^{I_K})$ (sitting inside $H^1(G_K, M^*)$ via inflation).
   (a) Show that $H^1_{\mathrm{ur}}(K, M)$ and $H^1_{\mathrm{ur}}(K, M^*)$ annihilate each other under the Tate pairing.
   (b) Show that $H^1_{\mathrm{ur}}(K, M)$ and $H^1_{\mathrm{ur}}(K, M^*)$ are the annihilators of each other under the Tate pairing. [Hint: enough to look at cardinalities.]
2. For $G_{\mathbb{R}} \cong \mathbb{Z}/2\mathbb{Z}$ and let M be a finite $G_{\mathbb{R}}$-module. Show that the pairing
$$H^1(G_{\mathbb{R}}, M) \times H^1(G_{\mathbb{R}}, M^*) \to H^2(G_{\mathbb{R}}, M \otimes M^*) \to H^2(G_{\mathbb{R}}, \mu_\infty(\mathbb{C})) \stackrel{\mathrm{inv}_{\mathbb{R}}}{\to} \frac{1}{2}\mathbb{Z}/\mathbb{Z}$$
is a perfect pairing.
3. Let $K/\mathbb{Q}_p$ be finite. Show that the natural map $K^\times \to \varprojlim K^\times/(K^\times)^n$ is injective.
4. (a) Let A be an abelian group and B a finite index subgroup such that $\cap A^n = \{1\}$ and $B = \varprojlim B/(B \cap A^n)$, where $A^n = \{a^n | a \in A\}$. Show that $A = \varprojlim A/A^n$.
   (b) Show that if $K/\mathbb{Q}_p$ is a finite extension then $\mathcal{O}_K^\times = \varprojlim \mathcal{O}_K^\times/(\mathcal{O}_K^\times)^n$.
5. Let G be a profinite group and M and N discrete G-modules. By an extension of N by M which is split as an abelian group we mean an exact sequence of G-modules
$$0 \to M \to E \to N \to 0$$
such that there exists a section $N \to E$ in the category of abelian groups, i.e., there exists an abelian group homomorphism $s : N \to E$ such that the composition $N \stackrel{s}{\to} E \to N$ is the identity. We call two such extensions E and E' equivalent if there exists an isomorphism $E \cong E'$ compatible with the injections $M \hookrightarrow E$ and $M \hookrightarrow E'$ and with the surjections $E \to N$ and $E' \to N$. Make $\mathrm{Hom}(N, M)$ into a G-module by setting $g(f) = g \circ f \circ g^{-1}$ for $g \in G$ and $g \in \mathrm{Hom}(N, M)$.
   (a) Show that there is a bijection between equivalence classes of extensions of N by M which split as abelian groups and $H^1(G, \mathrm{Hom}(N, M))$. [Hint: If $\phi \in Z^1(G, \mathrm{Hom}(N, M))$ define a G-module structure of $M \oplus N$ by setting
$$g(m \oplus n) = (gm + \phi(g)(n)) \oplus gn.]$$

(b) If N is a trivial G-module, show that the element in $H^1(G, \mathrm{Hom}(N, M))$ parametrising E is the image of $\delta_0(E) \in \mathrm{Hom}(N, H^1(G, M))$ under the natural map

$$\mathrm{Hom}(N, H^1(G, M)) \to H^1(G, \mathrm{Hom}(N, M))$$

which you should describe. Here $\delta_0 : N^G = N \to H^1(G, M)$ is the connecting homomorphism in the cohomology exact sequence.

6. Suppose that H is a subgroup of a finite group G and that ρ is a representation of H over a field K. Show that

$$\mathrm{Tr}\,\mathrm{Ind}_H^G(\rho)(g) = \sum_{k \in G/H} \mathrm{Tr}\,\rho(kgk^{-1})$$

where the sum is over those k for which $kgk^{-1} \in H$.