

# Math 160b “Syllabus”

Andrei Jorza

2012-01-03

## Contents

### 1 Fields with valuations

- 1.1 Recollections on valuations
  - 1.1.1 Definition and examples
  - 1.1.2 Integral local rings
- 1.2 Topology
  - 1.2.1 Completeness
  - 1.2.2 Hensel’s lemma
- 1.3 Newton polygons
  - 1.3.1 Valuations on polynomials
  - 1.3.2 Newton polygons definition
  - 1.3.3 Slopes and roots
  - 1.3.4 Newton polygons and products
  - 1.3.5 A Euclidean algorithm lemma
  - 1.3.6 Newton polygons and factorization
- 1.4 Applications to complete local fields
  - 1.4.1 Extensions of valuations
  - 1.4.2 Traces and complete local rings
  - 1.4.3 Valuations and the Galois group
- 1.5 Traces
  - 1.5.1 Perfectness of the trace pairing
  - 1.5.2 Extensions of valuations and traces
- 1.6 Extensions of discretely valued fields
  - 1.6.1 Lemma on extensions of dvrs
  - 1.6.2 Extensions of completions and the Galois group
- 1.7 Recollections on Galois theory
  - 1.7.1 Profinite Galois groups
  - 1.7.2 Main theorem of Galois theory
  - 1.7.3 Arithmetic and geometric Frobenius

### 2 Local fields

- 2.1 Basic properties
  - 2.1.1 Why study local fields
  - 2.1.2 Local fields notations
  - 2.1.3 Teichmüller lemma
  - 2.1.4 Power series expansion in terms of the uniformizer
- 2.2 Ramification of local fields
  - 2.2.1 Inertia and ramification indices, different
  - 2.2.2 Ramification and extensions

- 2.2.3 Different and minimal polynomial of uniformizer
- 2.2.4 Lemma on existence of maximal unramified subfield
- 2.2.5 Unramified extensions
- 2.2.6 Tame extensions

### 3 Galois groups of local fields

- 3.1 The ramification filtration
  - 3.1.1 Lower filtration
  - 3.1.2 Ramification function
  - 3.1.3 Upper filtration
  - 3.1.4 Solvability of extensions of local fields
  - 3.1.5 The function  $i_{L/K}$  and the ramification function
  - 3.1.6 Ramification filtration and different
- 3.2 Ramification in extensions
  - 3.2.1 The function  $i_{L/K}$  in extension towers
  - 3.2.2 Herbrand's theorem
  - 3.2.3 Upper filtration for infinite/non-Galois extensions
  - 3.2.4 Facts about upper filtration for infinite extensions
  - 3.2.5 Preview of Hasse-Arf
- 3.3 Traces and norms
  - 3.3.1 Lemma on traces in extensions
  - 3.3.2 Lemma on norms in extensions
  - 3.3.3 An exact sequence with norms and lower filtrations

### 4 Representation theory

- 4.1 Basics
  - 4.1.1 Recollections on algebra
  - 4.1.2 Brauer groups
  - 4.1.3 Lemma on representations of  $p$ -groups
- 4.2 Group representations
  - 4.2.1 Basics
  - 4.2.2 Inductions
  - 4.2.3 Frobenius reciprocity
  - 4.2.4 Mackey's theorem on inductions
  - 4.2.5 Grothendieck ring and Brauer's theorem
  - 4.2.6 Facts about representations
  - 4.2.7 Example of nonsemisimple representation
- 4.3 Galois representations
  - 4.3.1 The Weil group and its topology
  - 4.3.2 Conductor of a Galois representation
  - 4.3.3 The conductor is an integer
  - 4.3.4 Lemma on Galois representations and filtrations
  - 4.3.5 Conductor-discriminant formula

### 5 Group cohomology

- 5.1 Modules with group actions
  - 5.1.1 Lemma on profinite groups
  - 5.1.2 Modules with actions
  - 5.1.3 Induction as an exact functor
- 5.2 Recollections on homological algebra
  - 5.2.1 Basics
  - 5.2.2 Derived functors

- 5.2.3  $\delta$ -functors
- 5.2.4 Effaceability and universal  $\delta$ -functors
- 5.3 Group cohomology as a  $\delta$ -functor
  - 5.3.1 Modules with actions have enough injectives
  - 5.3.2 Group cohomology
  - 5.3.3 Shapiro's lemma
- 5.4 Operations on group cohomology
  - 5.4.1 Restriction
  - 5.4.2 Corestriction
  - 5.4.3 Inflation and properties
  - 5.4.4 Corestriction on cocycles
- 5.5 Group cohomology in terms of cochains
  - 5.5.1 Locally constant functions
  - 5.5.2 An acyclic resolution
  - 5.5.3 Cochains and cohomology
  - 5.5.4 Trivial group actions
  - 5.5.5 Cohomology and limits
  - 5.5.6 Two corollaries on cohomology
  - 5.5.7 Dimension shifting
  - 5.5.8 Cohomology of procyclic groups
- 5.6 Non-abelian cohomology
  - 5.6.1 Definitions
  - 5.6.2 Non-abelian  $H^1(G, -)$  as a functor
  - 5.6.3 Galois descent for semilinear representations
  - 5.6.4 Hilbert 90
- 5.7 Brauer groups
  - 5.7.1 Central simple algebras and the cohomology of PGL
  - 5.7.2 The Brauer group of finite extensions
  - 5.7.3 Kummer theory

## 6 Galois cohomology of local fields

- 6.1 Spectral sequences
  - 6.1.1 Basics
  - 6.1.2 The Hochschild-Serre spectral sequence and an example
  - 6.1.3 An exact sequence
  - 6.1.4 Inflation-restriction sequence
- 6.2 Galois cohomology of local fields
  - 6.2.1 Cohomology of inertia
  - 6.2.2 A corollary on the cohomology of torsion Galois modules
  - 6.2.3 The Brauer group of a local field and the Galois cohomology of its maximal unramified extension
  - 6.2.4 Cohomology of residue fields
  - 6.2.5 Cohomology and filtrations
  - 6.2.6 Cohomology of the ring of integers of the maximal unramified extension
  - 6.2.7 The valuation and cohomology
  - 6.2.8 Galois cohomology of the integers
- 6.3 The invariant map
  - 6.3.1 Construction
  - 6.3.2 Compatibility under extensions
  - 6.3.3 Torsion in the Brauer group

## 7 Local class field theory

- 7.1 Cup products
    - 7.1.1 Basics
    - 7.1.2 Properties
  - 7.2 Tate pairing
    - 7.2.1 Finiteness of Galois cohomology
    - 7.2.2 Tate dual
    - 7.2.3 Tate pairing
  - 7.3 Tate duality
    - 7.3.1 Lemma on dual of corestriction
    - 7.3.2 Tate duality and base change
    - 7.3.3 Reduction to trivial modules
    - 7.3.4 Proof in degrees 0 and 2
    - 7.3.5 Proof in degree 1
    - 7.3.6 Properties of the Tate pairing
  - 7.4 Artin reciprocity
    - 7.4.1 Double duals
    - 7.4.2 Construction of the reciprocity map
    - 7.4.3 Artin reciprocity and the norm
    - 7.4.4 A lemma on topology of local fields
    - 7.4.5 The main theorem on the Galois group of finite extensions
    - 7.4.6 Artin reciprocity and the transfer map
    - 7.4.7 Artin reciprocity and valuation
  - 7.5 The absolute Galois group
    - 7.5.1 Artin reciprocity and the Weil group
    - 7.5.2 A lemma on the lower filtration
    - 7.5.3 Hasse-Arf
    - 7.5.4 Compatibility of the Lie and Galois filtrations under the reciprocity map
    - 7.5.5 The maximal abelian extension
- 8 The Euler-Tate characteristic formula**
- 8.1 The main theorem
  - 8.2 Reductions
    - 8.2.1 Extension to the Grothendieck ring
    - 8.2.2 Reduction to vector spaces over  $\mathbb{F}_\ell$
    - 8.2.3 Behavior under induction
    - 8.2.4 Behavior under base-change
  - 8.3 The case  $\ell \neq p$ 
    - 8.3.1 Galois cohomology of inertia
    - 8.3.2 The proof of the Euler-Tate formula
  - 8.4 The structure of  $K^\times$ 
    - 8.4.1 A decomposition
    - 8.4.2 Finiteness of roots of unity
    - 8.4.3 log and exp
    - 8.4.4 Torsion free units
  - 8.5 The case  $\ell = p$ 
    - 8.5.1 Action of wild inertia
    - 8.5.2 Weight decomposition
    - 8.5.3 A lemma on weights
    - 8.5.4 Weights and base change
    - 8.5.5 Reduction to characters
    - 8.5.6 A lemma on units
    - 8.5.7 The proof of the Euler-Tate formula