Math 162b
Problem Set 1

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These exercises require two inputs from class field theory.

- The p-adic cyclotomic character $\chi_p : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ defined as the exponent such that for every p^k-th root of unity ζ and $g \in G_{\mathbb{Q}_p}$ one has $g\zeta = \zeta^{\chi_p(g)}$. Alternatively, one may use the inverse Artin map $\mathrm{rec}^{-1} : I_{\mathbb{Q}_p}^{\mathrm{ab}} \cong \mathbb{Z}_p^\times$ and define $\chi_p(\mathrm{Frob}_p^m \sigma) = \mathrm{rec}^{-1}(\sigma)$ where $\sigma \in I_{\mathbb{Q}_p}$.
- One has an isomorphism
$$G_{\mathbb{Q}}^{\mathrm{ab}} \cong \mathbb{A}_{\mathbb{Q}}^\times / \overline{\mathbb{Q}^\times(0,\infty)}$$

Exercise 1 (Results needed for Ax-Sen)**.** In this exercise you will study powers of p in binomial coefficients.

1. Show that $v_p((p^k n)!) = \dfrac{n(p^k-1)}{p-1} + v_p(n!)$.
2. Show that $v_p\left(\binom{p^{k+1}}{p^k}\right) = 1$.
3. Show that if $p \nmid n$ then $v_p\left(\binom{p^k n}{p^k}\right) = 0$.

Exercise 2 (Stable lattices)**.** Let G be a profinite group and let $\rho : G \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ be a continuous Galois representation. The purpose of this exercise is to show that a conjugate of ρ takes values in $\mathrm{GL}(n, \mathcal{O}_L)$ for some finite extension $L/\mathbb{Q}_\ell$ (thus one can reduce modulo ℓ).

1. Show that $\operatorname{Im}\rho$ is compact Hausdorff.
2. Show that there exist finite extensions $L_k/\mathbb{Q}_\ell$ such that $\operatorname{Im}\rho = \bigcup_k (\operatorname{Im}\rho \cap \mathrm{GL}(n, L_k))$.
3. Use the Baire category theorem to deduce that for some k, the interior U of $\operatorname{Im}\rho \cap \mathrm{GL}(n, L_k)$ is not empty.
4. Show that $\operatorname{Im}\rho/U$ is a finite coset space and show that $L = L_k$ may be chosen such that these cosets are defined over L.
5. Take the average of $\mathcal{O}_L^n$ over these cosets to deduce the existence of a full rank lattice stable under G, i.e., that a conjugate of ρ lands in $\mathrm{GL}(n, \mathcal{O}_L)$.

Exercise 3 (Weil-Deligne representations)**.** Let $K/\mathbb{Q}_p$ be a finite extension and $\ell \neq p$. Recall that an ℓ-adic Weil-Deligne representation of W_K is a pair (r, N) of a continuous Galois representation $r : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ (recall that continuity is with respect to the topology on the Weil group W_K, i.e., r is a homomorphism with open kernel) and a linear map $N \in \mathrm{End}(r)$ such that $r(g)Nr(g)^{-1} = |\,\mathrm{rec}_K^{-1}(g)|N$, where $\mathrm{rec}_K : K^\times \cong W_K^{\mathrm{ab}}$ is the Artin reciprocity map. The Weil-Deligne representation (r, N) is said to be bounded if the eigenvalues of all $r(g)$ are ℓ-adic units. The point of this exercise is to show that if $\ell \neq p$ then continuous ℓ-adic representations of the full Galois group G_K are the same as bounded Weil-Deligne representations of W_K.

1. Show that N is a nilpotent matrix.

2. Let $t : I_K \to I_K/P_K \cong \mathrm{Gal}(K^{\mathrm{t}}/K^{\mathrm{ur}}) \cong \prod_{q \neq p} \mathbb{Z}_q(1) \to \mathbb{Z}_\ell(1) \approx \mathbb{Z}_\ell$ given by $\sigma(\varpi_K^{1/n}) \equiv t(\sigma)\varpi_K^{1/n} \pmod{n}$. Show that t is well-defined up to $\mathbb{Z}_\ell^\times$.

3. Let ϕ be a lift to G_K of the geometric Frobenius $\mathrm{Frob}_K \in G_{K^{\mathrm{ur}}/K} \cong G_{k_K}$ (well-defined up to conjugacy); you may assume (from 160b) that $t(\phi^n \sigma \psi^{-n}) = q_K^{-n} t(\sigma)$, and that $\mathrm{rec}_K^{-1}(\phi)$ is a uniformizer. Given a bounded Weil-Deligne representation (r, N) define

$$\rho(\phi^n \sigma) = r(\phi^n \sigma) \exp(t(\sigma) N)$$

for $n \in \mathbb{Z}$ and $\sigma \in I_K$.

   (a) Show that if $q_K = \# k_K$ then $\exp(t(\sigma)N) r(\phi^n \sigma) = r(\phi^n \sigma) \exp(t(\sigma) q_K^n N)$.
   (b) Show that $\rho(\phi^n \sigma)\rho(\phi^m \tau) = \rho(\phi^n \sigma \phi^m \tau)$.
   (c) Show that ρ extends to a continuous ℓ-adic representation $\rho : G_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ (you may assume here that (r, N) being bounded is equivalent to $\mathrm{Im}\, r$ having compact closure).
   (d) Show that the isomorphism class of ρ is independent of the choice of ϕ or t.

4. Conversely, start with a continuous ℓ-adic representation $\rho : G_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$. You may assume Grothendieck's ℓ-adic monodromy theorem, that ρ is potentially unramified, i.e., there exists a finite extension L/K such that $\rho|_{G_L}$ is unramified, or equivalently that $\rho(I_L) = I_n$.

   (a) Let $L/\mathbb{Q}_\ell$ be a finite extension such that (a suitably chosen conjugate of) ρ takes values in $\mathrm{GL}(n, \mathcal{O}_L)$. Consider the group $H = \rho^{-1}(1 + \ell^2 M_{n \times N}(\mathcal{O}_L)) \cap I_K$. Show that H is an open normal pro-ℓ group.
   (b) Deduce that $\rho : H \to 1 + \ell^2 M_{n\times N}(\mathcal{O}_L)$ extends to $\widetilde{\rho} : \mathbb{Z}_\ell \to 1 + \ell^2 M_{n \times N}(\mathcal{O}_L)$.
   (c) Show that $t(H) = \ell^s \mathbb{Z}_\ell$ for some s (ℓ-adic monodromy!) and write $N = \log(\widetilde{\rho}(\ell^s))\ell^{-s}$.
   (d) Show that for $h \in H$, $\rho(h) = \exp(t(h)N)$.
   (e) Show that $r(\phi^n \sigma) = \rho(\phi^n \sigma)\exp(-t(\sigma)N)$ is a homomorphism with open kernel.
   (f) Show that (r, N) is a Weil-Deligne representation.
   (g) Show that $\mathrm{Im}\, r \subset \mathrm{Im}\, \rho \exp(\mathbb{Z}_\ell N)$ and thus that (r, N) is bounded.

Exercise 4 (Compatible systems for Hecke characters)**.** Let $\psi : \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be a Hecke character such that the infinite component ψ_∞ in $\psi = \psi_\infty \otimes \bigotimes_p \psi_p$ is the character $\psi_\infty(x) = |x|^n$ or $\mathrm{sign}(x)|x|^n$ for an integer n. (You can think of these as algebraic automorphic representations on GL(1).)

1. Show that for every prime ℓ there exists a (continuous) character $\eta_\ell : \mathbb{Q}_\ell^\times \to \overline{\mathbb{Q}}_\ell^\times$ such that $\eta_\ell(x) = \psi_\infty(x)$ for $x \in \mathbb{Q}^\times$.

2. Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. Show that the character $\Psi_\ell = \psi\psi_\infty^{-1}\eta_\ell : \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ is continuous, vanishes on $\mathbb{Q}^\times$ and $(0, \infty)$ and deduce that it induces a continuous ℓ-adic Galois representation $\Psi_\ell : G_\mathbb{Q} \to \mathrm{GL}(1, \overline{\mathbb{Q}}_\ell)$.

3. Show that if p is a prime such that ψ_p is unramified (i.e., $\psi_p(\mathbb{Z}_p^\times) = 1$) and if ℓ and ℓ' are two primes different from p then $\Psi_\ell|_{G_{\mathbb{Q}_p}}$ and $\Psi_{\ell'}|_{G_{\mathbb{Q}_p}}$ are unramified and act identically on Frob_p.

4. Show that if ψ_p is unramified then $\Psi_p|_{G_{\mathbb{Q}_p}}$ is of the form χ_p^n times an unramified character. (These are the prototypical "crystalline" representations and the integer n is the "Hodge-Tate weight"; note that it has no chance of being unramified, since the cyclotomic character is very ramified; in general, the local p-adic Galois representation attached to an automorphic representation which is unramified at p will be crystalline with Hodge-Tate weights dictated by the component at infinity.)