Math 162b
Problem Set 2

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Exercise 1. *Let $K/F = \mathbb{Q}_p$ be a finite extension and let $K_n = K(\zeta_{p^n})$ and $F_n = F(\zeta_{p^n})$. Recall that we showed in class that $p^n v_p(\mathcal{D}_{K_n/F_n})$ is bounded. Show that there exists a constant c and a bounded sequence a_n such that*

$$v_p(\mathcal{D}_{K_n/F}) = n + c + \frac{a_n}{p^n}$$

[Hint: recall that $G^u_{F_n/F} = G_{F_n/F_{\lfloor u \rfloor}}$.]

Exercise 2. *In this exercise you will show that $\overline{\mathbb{Q}_p}$ is not complete, i.e., that $\mathbb{C}_p \neq \overline{\mathbb{Q}_p}$.*

1. *Show that one may find roots of unity a_n such that*
   - (a) $a_n \in \mathbb{Q}_p^{\mathrm{ur}}$,
   - (b) $a_{n-1} \in \mathbb{Q}_p(a_n)$ *and*
   - (c) $[\mathbb{Q}_p(a_n) : \mathbb{Q}_p(a_{n-1})] > n$.
2. *Show that* $\alpha = \sum_{n=1}^{\infty} a_n p^n \in \mathbb{C}_p$.
3. *If $\alpha \in \overline{\mathbb{Q}_p}$ let $m = [\mathbb{Q}_p(\alpha) : \mathbb{Q}_p]$, $\alpha_m = \sum_{n=1}^{m} a_n p^n$. Show that there exists a Galois extension $M/\mathbb{Q}_p$ such that $\alpha, \alpha_m, a_m \in M$ and that there exist $\sigma_1, \ldots, \sigma_{m+1} \in \mathrm{Gal}(M/\mathbb{Q}_p(a_{n-1}))$ such that $\sigma_1(a_m), \ldots, \sigma_{m+1}(a_m)$ are all distinct.*
4. *Show that $\sigma_i(a_m)$ and $\sigma_j(a_m)$ for $i \neq j$ are distinct modulo p and deduce that $v_p(\sigma_i(a_m) - \sigma_j(a_m)) = 0$.*
5. *Deduce that $v_p(\sigma_i(\alpha_m) - \sigma_j(\alpha_m)) = m$.*
6. *Show that $v_p(\sigma_i(\alpha) - \sigma_i(\alpha_m)) \geq m + 1$.*
7. *Deduce that $v_p(\sigma_i(\alpha) - \sigma_j(\alpha)) = m$.*
8. *Conclude that the $\sigma_i(\alpha)$ are distinct and derive a contradiction.*

Exercise 3. *Let $K/\mathbb{Q}_p$ be a finite extension.*

1. *Let L/K be an algebraic extension. We've seen that $\widehat{L} \neq L$ in the previous exercise. Show that $\widehat{L} \cap \overline{K} = L$.*
2. *Let M/K_∞ be a finite extension. Show that there exists a finite extension L/K such that $M = L_\infty$.*

Exercise 4. *Let p be a prime.*

1. *Show that* $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ *converges in the range* $|x| < |p|^{\frac{1}{p-1}}$.

2. *Show that* $\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ *converges in the range* $|x| < 1$.

3. *Show that one gets a* G_K*-equivariant homomorphism* $\log_{\overline{\mathbb{Z}}_p^\times} : \overline{\mathbb{Z}}_p^\times \to \overline{\mathbb{Z}}_p$.

4. *Show that for every* $c \in \overline{\mathbb{Q}}_p$ *there exists a unique* G_K*-equivariant homomorphism* $\log_{\overline{\mathbb{Q}}_p^\times} : \overline{\mathbb{Q}}_p^\times \to \overline{\mathbb{Q}}_p$ *such that* $\log_{\overline{\mathbb{Q}}_p^\times} |_{\overline{\mathbb{Z}}_p^\times} = \log_{\overline{\mathbb{Z}}_p^\times}$ *and* $\log_{\overline{\mathbb{Q}}_p^\times}(p) = c$.