Math 162b
Problem Set 3

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Exercise 1. *Show that if A is a strict p-ring with perfect residue ring R and if $[\cdot] : R \to A$ is the Teichmüller lift then every element $\alpha \in A$ can be written uniquely as*

$$\alpha = \sum_{n \geq 0} p^n [\alpha_n]$$

for $\alpha_n \in R$.

Exercise 2. *Let $\overline{S} = \mathbb{F}_p[X_i^{p^{-m}}, Y_i^{p^{-m}}]_{i,m \geq 0}$, $S = \mathbb{Z}_p[X_i^{p^{-m}}, Y_i^{p^{-m}}]_{i,m \geq 0}$ and $\widehat{S} = \varprojlim S/p^n S$.*

1. *If $\overline{S}_i \in \overline{S}$ such that*
$$\sum_{n \geq 0} p^n [X_n] + \sum_{n \geq 0} p^n [Y_n] = \sum_{n \geq 0} p^n [\overline{S}_n]$$
in $\widehat{S}$ show that $\overline{S}_n$ is a homogeneous polynomial of degree 1 in $X_0, \ldots, X_n$, and a homogeneous polynomial of degree 1 in $Y_0, \ldots, Y_n$.

2. *If $\overline{P}_i \in \overline{S}$ such that*
$$\sum_{n \geq 0} p^n [X_n] \sum_{n \geq 0} p^n [Y_n] = \sum_{n \geq 0} p^n [\overline{P}_n]$$
in $\widehat{S}$ show that $\overline{P}_n$ does not contain monomials with only X-s or only Y-s.

Exercise 3. *Let $\mathrm{D}_{\mathrm{HT},K} : \mathrm{Rep}_{\mathbb{C}_p}(G_K) \to \mathrm{GrVec}_K$. Show that if L/K is a finite extension then $\mathrm{D}_{\mathrm{HT},L} \cong \mathrm{D}_{\mathrm{HT},K} \otimes_K L$.*

Exercise 4. *Consider $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ the two dimensional representation with the action*

$$g \mapsto \begin{pmatrix} 1 & \log \chi_{\mathrm{cycl}}(g) \\ 0 & 1 \end{pmatrix}$$

Show that $\Theta_{\mathrm{Sen}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Exercise 5. *Let $\mathrm{B}^n_{\mathrm{Sen}} \subset \mathbb{C}_p[\![T]\!]$ be the set of power series with radius of convergence at least p^{-n}. The Galois group G_K acts on $\mathbb{C}_p[\![T]\!]$ semilinearly via $g(T) = T + \log \chi_{\mathrm{cycl}}(g)$.*

1. *Show that $\mathrm{B}^n_{\mathrm{Sen}}$ is stable under the action of G_{K_n}.*

2. *If $f = \sum_{k \geq 0} a_k T^k \in (\mathrm{B}^n_{\mathrm{Sen}})^{G_{K_n}}$ show that $a_k \in \widehat{K_\infty}$ for all $k \geq 0$.*

3. *Show that for* f *as above and any* $g \in G_{K_n}$ *we have*

$$a_i = \sum_{k \geq i} \binom{k}{i} g^{-1}(a_k)(-\log \chi_{\mathrm{cycl}}(g))^{k-i}$$

4. *Deduce that*

$$g(\mathrm{pr}_m(a_i)) = \sum_{k \geq i} \binom{k}{i} \mathrm{pr}_m(a_k)(-\log \chi_{\mathrm{cycl}}(g))^{k-i}$$

5. *Show that the left hand side is a locally constant function of* g.

6. *Show that the right hand side is an analytic function of* g.

7. *Deduce that both sides of the equality are constant, equal to 0 when* $i \geq 1$.

8. *Conclude that* $(\mathrm{B}^n_{\mathrm{Sen}})^{G_{K_n}} = K_n$.

9. *For* $V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ *let* $e_1, \ldots, e_d$ *be the* K_n *basis of* $\mathrm{D}_{\mathrm{Sen}}(V)$ *which descends to* K_n. *Show that* $f_i = \exp(-T\Theta_{\mathrm{Sen}})e_i \in \mathrm{B}_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V := \varinjlim_n \mathrm{B}^n_{\mathrm{Sen}} \otimes_{\mathbb{C}_p} V$.

10. *Prove that the* K_∞*-linear map* $\iota(e_i) = f_i$ *gives* $\mathrm{D}_{\mathrm{Sen}}(V) = \mathrm{D}_{\mathrm{B}_{\mathrm{Sen}}} := \varinjlim_n \mathrm{D}_{\mathrm{B}^n_{\mathrm{Sen}}}(V)$ *as* K_∞ *vector spaces.*

11. *Show that* $-\dfrac{d}{dT}\iota(v) = \iota(\Theta_{\mathrm{Sen}}(v))$ *and thus that* $\mathrm{D}_{\mathrm{B}_{\mathrm{Sen}}} : \mathrm{Rep}_{\mathbb{C}_p}(G_K) \to \mathcal{S}_{K_\infty}$ *is the same as* $\mathrm{D}_{\mathrm{Sen}}$.