MATH 162B
PROBLEM SET 4

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This problem set has several sections, and some definitions.

Contents

1. Divided powers

Definition 1 (Divided powers)**.** Divided powers are a device which allows one to make sense of $x^n/n!$ even in positive characteristic.

Let A be a commutative ring and I and ideal of A. A divided power structure (a PD structure, for puissances divisées) on I is a collection of maps $\gamma_i : I \to A$ for $i \geq 0$ such that

(i) For all $x \in I$ we have $\gamma_0(x) = 1$, $\gamma_1(x) = x$ and $\gamma_i(x) \in I$ for $i \geq 2$.
(ii) For all $x, y \in I$ we have
$$\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$$
(iii) For $\lambda \in A$ and $x \in I$ we have $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$.
(iv) For $x \in I$ we have
$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$$
(v) For $x \in I$ we have
$$\gamma_p(\gamma_q(x)) = \frac{(pq)!}{p!(q!)^p}\gamma_{pq}(x)$$

In this case we say that (A, I, γ) is a PD ring.

Exercise 1. (1) Show that $n!\gamma_n(x) = x^n$ for $x \in I$ and $n \geq 1$ and deduce that every ideal of a $\mathbb{Q}$-algebra has a unique PD structure.
(2) Let $K/\mathbb{Q}_p$ be a finite extension, let $\mathcal{O}_K$ be the ring of integers and π_K be a uniformizer. Let $e = e_{K/\mathbb{Q}_p}$ be the ramification index in which case $v_K(p) = e$ where $v_K(\pi_K) = 1$. Show that the maximal ideal of $\mathcal{O}_K$ has a PD structure if and only if $e \leq p-1$.
(3) Let (A, I, γ) be a PD ring such that I is principal. Show that for any A-algebra B the ideal IB has a PD structure which restricts to γ on $I \subset IB$.
(4) Let k be a ring of characteristic 2 and let
$$A = k[x_1, \ldots, x_6]/(x_1^2, \ldots, x_6^2, x_1x_2 + x_3x_4 + x_5x_6)$$
Show that the ideal $I = (x_1, \ldots, x_6)$ has no PD structure as follows:
  (a) Show that $\gamma_2(x_ix_j) = 0$.
  (b) Show that $\gamma_2(x_1x_2 + x_3x_4) = 0 = x_1x_2x_3x_4$ and conclude that no γ can exist.

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2. Towards the fundamental sequence for $\mathrm{B}_{\mathrm{cris}}$

Exercise 2. This exercise feeds into the proof of the fundamental exact sequence for $\mathrm{B}_{\mathrm{cris}}$. For a positive integer n let $n = q(n)(p-1) + r(n)$ where $0 \leq r(n) < p-1$ and write

$$t^{\{n\}} = t^{r(n)}\gamma_{q(n)}\left(\frac{t^{p-1}}{p}\right)$$

Define $K_0 = \mathrm{W}(k_K)[1/p]$ and $\Lambda_\varepsilon \subset K_0[\![t]\!]$ be the set of power series that can be written as

$$\sum_{n\geq 0} a_n t^{\{n\}}$$

where $a_n \in \mathrm{W}(k_K)$ converge p-adically to 0. Recall that we have defined $I^{[r]}\,\mathrm{A}_{\mathrm{cris}} = \cap\varphi^{-n}(\mathrm{Fil}^r\,\mathrm{A}_{\mathrm{cris}})$.

(1) S_ε and Λ_ε.
   (a) Show that Λ_ε is a φ-stable and G_K-stable subring of $K_0[\![t]\!]$.
   (b) Show that $[\varepsilon] - 1 \in \Lambda_\varepsilon$.
   (c) Show that S_ε defined as $\mathrm{W}(k_K)[\![[\varepsilon]-1]\!]$ is a G_K-stable and φ-stable sub-$\mathrm{W}(k_K)$-algebra of Λ_ε.

(2) $I^{[r]}\,\mathrm{A}_{\mathrm{cris}}$.
   (a) Write $I(r) = \{\sum_{n\geq r} a_n t^{\{n\}} | a_n \in \mathrm{W}(\mathrm{R}), a_n \to 0 \text{ } p\text{-adically}\}$. Show that $I(r) \subset I^{[r]}$.
   (b) Show that $I^{[0]} \subset I(0)$.
   (c) If $a = \sum_{n\geq r-1} a_n t^{\{n\}} \in I^{[r]}$ show that $a_{r-1}t^{\{r-1\}} \in I^{[r]}$.
   (d) Show that $a_{r-1} \in I^{[1]}\,\mathrm{W}(\mathrm{R})$ and deduce that $a_{r-1}t^{\{r-1\}} \in ([\varepsilon]-1)t^{\{r-1\}}\,\mathrm{A}_{\mathrm{cris}} = t \cdot t^{\{r-1\}}\,\mathrm{A}_{\mathrm{cris}}$.
   (e) Deduce that $I^{[r]} = I(r)$.

Exercise 3. This exercise is used in studying congruences between Galois representations. In this exercise you will show that for every r there exists λ such that for all $m \geq \lambda$

$$\mathrm{A}_{\mathrm{cris}} \cap p^m t^{-r}\,\mathrm{A}_{\mathrm{cris}} \subset \sum_{i+j=m-\lambda} p^i I^{[j]}\,\mathrm{A}_{\mathrm{cris}}$$

You may assume that $\mathrm{A}_{\mathrm{cris}}/I^{[r]}\,\mathrm{A}_{\mathrm{cris}}$ has no p-torsion.

(1) Let $c_n = \dfrac{p^{q(n+r)}q(n+r)!}{p^{q(n)}q(n)!}$. Show that $\lambda = -\min(n - v_p(c_n))$ is a nonnegative integer.
(2) Show that if $a = \sum_{n\geq 0} a_n t^{\{r\}}$ then $t^r a = \sum_{n\geq 0} a_n c_n t^{\{n+r\}}$.
(3) Suppose $a = \sum_{n\geq 0} a_n t^{\{r\}} \in \mathrm{A}_{\mathrm{cris}} \cap p^m t^{-r}\,\mathrm{A}_{\mathrm{cris}}$. Such a power series representation is not unique, and this subpart will show by induction that one can arrange the power series such that whenever $0 \leq n \leq m - \lambda$ we have $a_n \in p^{m-\lambda-n}\,\mathrm{W}(\mathrm{R})$ and $p^m \mid a_n c_n$.
   (a) Show that we may write $t^r a = p^m \sum_{n\geq 0} b_n t^{\{n+r\}} \in p^m\,\mathrm{A}_{\mathrm{cris}}$. [Hint: Look modulo $I^{[r]}\,\mathrm{A}_{\mathrm{cris}}$.]
   (b) Show that $a_0c_0 - p^m b_0 \in p^{v_p(c_0)}I^{[1]}\,\mathrm{A}_{\mathrm{cris}}$ and deduce the base case of the induction. [Hint: Look modulo $I^{[r+1]}\,\mathrm{A}_{\mathrm{cris}}$.]
   (c) Show the inductive step by repeating the argument from the base case for

$$\sum_{n\geq k} a_n c_n t^{\{n+r\}} = p^m\left(\sum_{n\geq 0} b_n t^{\{n+r\}} - \sum_{n=0}^{k-1}\frac{a_n c_n}{p^m}t^{\{n+r\}}\right)$$

   (d) Deduce the main statement.

3. Frobenius eigenspaces on $\mathrm{B}_{\mathrm{cris}}$

This rather long but computational exercise is used in studying analytically varying p-adic Galois representations.

Exercise 4. For $\varepsilon \in (0,1)$ let $\mathrm{A}_{\mathrm{cris}}\langle \varepsilon T, T]\!]$ be the set of power series in $\mathrm{A}_{\mathrm{cris}}[\![T, T^{-1}]\!]$ of the form $\sum_{n\in\mathbb{Z}} a_n T^n$ such that $\lim_{n\to\infty} |a_n|\varepsilon^{-n} = 0$. Here $|\cdot|$ on $\mathrm{B}^+_{\mathrm{cris}}$ is the p-adic norm.

(1) $\mathrm{A}_{\mathrm{cris}}\langle \varepsilon T, T]\!]$.

(a) Show that $\mathrm{A}_{\mathrm{cris}}\langle \varepsilon T, T]\!]$ is a ring.

(b) If $x \in \mathrm{R}$ such that $v_{\mathrm{R}}(x) > 0$ show that

$$F(x,T) := \sum_{n\in\mathbb{Z}} \varphi^n([x])T^n \in \mathrm{A}_{\mathrm{cris}}\langle \varepsilon T, T]\!]$$

for any $\varepsilon \in (0,1)$.

(c) Show that if $f \in \mathrm{A}_{\mathrm{cris}}\langle \varepsilon T, T]\!]$ then $f(\lambda)$ converges whenever $|\lambda^{-1}| < 1 < |\lambda| < \varepsilon^{-1}$. Deduce that $F(x,p^{-1})$ converges for all $x \in \mathfrak{m}_{\mathrm{R}}$.

(d) Show that if $x \in \mathcal{O}_{\mathbb{C}_p}$ such that $v_p(x) > 0$ then $\varphi(F(\widetilde{x},p^{-1})) = pF(\widetilde{x},p^{-1})$ and deduce that $F(x,p^{-1}) \in \mathrm{B}^{+,\varphi=p}_{\mathrm{cris}}$. Denote by L the closure inside $\mathrm{B}^{+,\varphi=p}_{\mathrm{cris}}$ of $\{F(\widetilde{x},p^{-1}) | x \in \mathcal{O}_{\mathbb{C}_p}, v_p(x) > 0\}$.

(2) $\mathrm{B}^{+,\varphi=p}_{\mathrm{cris}}$. Assume $p > 2$. For $s \in \mathcal{O}_{\mathbb{C}_p}$ we will write $\widetilde{s} = (s, s^{1/p}, \ldots)$ a lift to R.

(a) Show that $\mathrm{Fil}^1\,\mathrm{B}^{+,\varphi=p}_{\mathrm{cris}} = \mathbb{Q}_p \cdot t$. [Hint: use the fundamental exact sequence.]

(b) Let $x \in \mathrm{R}$.

(i) If $v_{\mathrm{R}}(x) > 1/(p-1)$ show that

$$\theta(F(x,p^{-1})) \equiv x^{(0)} \pmod{p^{pv_{\mathrm{R}}(x)-1}}$$

(ii) If $v_{\mathrm{R}}(x) < 1/(p-1)$ show that

$$\theta(p^{-1}F(\widetilde{p}x,p^{-1})) \equiv x^{(0)} \pmod{p^{p^{-1}(v_{\mathrm{R}}(x)+1)}}$$

(c) Let $u \in \mathbb{C}_p$ such that $v_p(u) \le 1/(p-1)$ and let s be a root of $x^p + px - pu = 0$.

(i) Show that $v_p(s) = p^{-1}(v_p(u)+1)$.

(ii) If moreover $v_p(u) \ge 1/(2(p-1))$ then for s as above we have

$$\theta(F(\widetilde{s},p^{-1})) \equiv u \pmod{p}$$

(d) Let $u \in \mathcal{O}_{\mathbb{C}_p}$ be a lift of a nonzero element of $\mathcal{O}_{\mathbb{C}_p}/(p)$ such that $v_p(u) > 1/(p-1)$. Construct a sequence $u_n \in \mathbb{C}_p$ such that $u_0 = u$ and $u_{n+1} = u_n - \theta(F(\widetilde{u_n},p^{-1}))$.

(i) Show that $v_p(u_n)$ is increasing with $\lim_{n\to\infty} v_p(u_n) = \infty$.

(ii) Deduce that for $n >> 0$ (for which $v_p(u_n) \ge 1$) one has $u \equiv \sum_{k=0}^{n} \theta(F(u_k,p^{-1})) \pmod{p}$.

(e) Let $u \in \mathcal{O}_{\mathbb{C}_p}$ be a lift of a nonzero element of $\mathcal{O}_{\mathbb{C}_p}/(p)$ such that $v_p(u) < 1/(p-1)$. Construct a sequence $u_n \in \mathbb{C}_p$ such that $u_0 = u$ and $u_{n+1} = u_n - \theta(p^{-1}F(\widetilde{pu_n},p^{-1}))$.

(i) Show that $v_p(u_n)$ is increasing with $\lim_{n\to\infty} v_p(u_n) = 1/(p-1)$.

(ii) For $n >> 0$ (such that $v_p(u_n) > 1/(2(p-1))$ show that

$$u \equiv \theta(F(\widetilde{s},p^{-1})) + \sum_{k=0}^{n} \theta(p^{-1}F(\widetilde{pu_k},p^{-1}))$$

where s is obtained from u_{n+1} which satisfies $1/(p-1) > v_p(u_{n+1}) > 1/(2(p-1))$.

(f) See part 1d for the definition of L.

(i) Show that the map $\theta : L \cap \theta^{-1}(\mathcal{O}_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}/(p)$ is surjective.

(ii) Deduce that the map $\theta : L \to \mathbb{C}_p$ is surjective.

(g) We now show that $L = \mathrm{B}^{+,\varphi=p}_{\mathrm{cris}}$.

(i) Let $h \in L$ such that $\theta(h) = 1 \in \mathbb{C}_p$. Show that h cannot be $G_{\mathbb{Q}_p}$-invariant. Let $g \in G_{\mathbb{Q}_p}$ such that $g(h) \ne h$.

(ii) Show that $g(h) - h \in \mathrm{Fil}^1\,\mathrm{B}^{+,\varphi=p}_{\mathrm{cris}}$.

(iii) Deduce that $\mathbb{Q}_p \cdot t \subset L$.

(iv) For $x \in \mathrm{B}^{+,\varphi=p}_{\mathrm{cris}}$ show there exists $y \in L$ such that $x - y \in \ker\theta$.

(v) Deduce there exists $z \in \mathrm{B}^{+,\varphi=1}_{\mathrm{cris}} = \mathbb{Q}_p$ such that $x - y = tz$ and thus $x \in L$.