MATH 162B
PROBLEM SET 5

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Exercise 1. This exercise will give an example[1] of an element in $\mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}} - \mathrm{B}^+_{\mathrm{cris}}$.

(1) Let $y = \dfrac{[\varepsilon^{1/p}] - 1}{[\varepsilon^{1/p^2}] - 1}$. Show that $y \in \mathrm{B}_{\mathrm{cris}}$ and $\dfrac{1}{y} \in \mathrm{B}^+_{\mathrm{dR}}$.
(2) Show that
$$\frac{1}{y} = \frac{\varphi(y)([\varepsilon^{1/p^2}] - 1)}{[\varepsilon] - 1} \in \mathrm{B}_{\mathrm{cris}}$$
(3) Show that $\dfrac{1}{\varphi(y)} \notin \mathrm{B}^+_{\mathrm{dR}}$ and deduce that $\dfrac{1}{y} \in \mathrm{Fil}^0 \mathrm{B}_{\mathrm{cris}} - \mathrm{B}^+_{\mathrm{cris}}$.

Exercise 2. Let $p \equiv 3 \pmod 4$ be a prime and let $\mathrm{Fil}^{-1} D = D = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2 \supset \mathrm{Fil}^0 D = \mathbb{Q}_p e_1 \supset \mathrm{Fil}^1 D = 0$ be a filtered φ-module with $\varphi(e_1) = -p^{-1} e_2$ and $\varphi(e_2) = e_1$.

(1) Show that D is irreducible.
(2) Show that $K = \mathbb{Q}_p(\sqrt{-1})$ is unramified over $\mathbb{Q}_p$, of degree 2.
(3) Show that $\widetilde{D} := D \otimes_{\mathbb{Q}_p} (K \otimes_{\mathbb{Q}_p} K)$ is reducible.
(4) If $D = \mathrm{D}_{\mathrm{cris}}(V)$ for a crystalline representation V show that $\widetilde{D} = \mathrm{D}_{\mathrm{cris},K}(V \otimes_{\mathbb{Q}_p} K)$.

Exercise 3. Let[2] $x_n = \dfrac{\omega^{p^n - 1}}{(p^n - 1)!}$.

(1) Show that $\lim_{n\to\infty} \omega x_n = 0$ in $\mathrm{B}^+_{\mathrm{cris}}$ but (x_n) does not converge in $\mathrm{B}^+_{\mathrm{cris}}$.
(2) Show that $\lim_{n\to\infty} x_n = 0$ in $\mathrm{B}_{\mathrm{cris}}$, and thus that the topology on $\mathrm{B}^+_{\mathrm{cris}}$ is not the subspace topology from $\mathrm{B}_{\mathrm{cris}}$.

Exercise 4. Let $K/\mathbb{Q}_p$ be a finite extension and let ϖ_K be a uniformizer. Let $\underline{\varpi} = (\varpi_K, \varpi_K^{1/p}, \ldots)$ and let $g(\underline{\varpi}) = \underline{\varpi}\varepsilon^{c(g)}$. Let $E(u)$ be the minimal polynomial of ϖ_K over K_0. Let $S = \varprojlim_m \left(\mathrm{W}(k_K) \left[u, \dfrac{E(u)^n}{n!} \right]_{n \geq 1} \right) / (p^m)$. Define:

- $\varphi(\sum a_n u^n) = \sum \sigma_{K_0}(a_n) u^{pn}$,
- $N = -u \dfrac{d}{du}$,
- $\mathrm{Fil}^i S = \left\{ \sum_{n \geq i} a_n \dfrac{E(u)^n}{n!} | a_n \in \mathrm{W}(k_K), a_n \to 0 \right\}$.

(1) Show that $N\varphi = p\varphi N$.
(2) Show that $\varphi(\mathrm{Fil}^i S) \subset p^i S$ for $0 \leq i \leq p - 1$.
(3) Show that $p^{-1}\varphi(E(u)) \in S^\times$.

Exercise 5. Recall the notation from Exercise 4. Let $\widehat{\mathrm{A}_{\mathrm{st}}} = \varprojlim_m \left(\mathrm{A}_{\mathrm{cris}} \left[\dfrac{X^n}{n!} \right]_{n \geq 1} \right) / (p^m)$. Define:

- $\varphi(\sum a_n X^n) = \sum \varphi(a_n)((1 + X)^p - 1)^n$,

Date: due Wednesday, March 14, 2012.
[1] I learned about this example from Laurent Berger.
[2] example due to Pierre Colmez

- $N = (1+X)\dfrac{d}{dX}$,
- for $g \in G_K$ let $g(\sum a_n X^n) = \sum g(a_n) g(X)^n$ where $g(X) = [\varepsilon]^{c(g)}(1+X) - 1$,
- $\operatorname{Fil}^i \widehat{\mathrm{A}_{\mathrm{st}}} = \left\{ \sum_{n \geq 0} a_n \dfrac{X^n}{n!} \in \widehat{\mathrm{A}_{\mathrm{st}}} | a_n \in \operatorname{Fil}^{i-n} \mathrm{A}_{\mathrm{cris}}, a_n \to 0 \right\}$.

(1) Show that the above formula gives an action of G_K on $\widehat{\mathrm{A}_{\mathrm{st}}}$.
(2) Show that $N\varphi = p\varphi N$.
(3) Show that $\dfrac{[\varpi]}{1+X} \in \widehat{\mathrm{A}_{\mathrm{st}}}$ and $u \mapsto \dfrac{[\varpi]}{1+X}$ gives a map $S \to \widehat{\mathrm{A}_{\mathrm{st}}}$.
(4) Show that this map $S \to \widehat{\mathrm{A}_{\mathrm{st}}}$ respects φ, N and filtrations.
(5) Show that the image of S in $\widehat{\mathrm{A}_{\mathrm{st}}}$ is contained in $\widehat{\mathrm{A}_{\mathrm{st}}}^{G_K}$. (Remark: one can show $S \cong \widehat{\mathrm{A}_{\mathrm{st}}}^{G_K}$).